Two-Scale Homogenization of Systems of Nonlinear Parabolic Equations

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Abstract

The aim of this thesis is to derive homogenization results for two different types of systems of nonlinear parabolic equations, namely reaction-diffusion systems involving different diffusion length scales and Cahn–Hilliard-type equations. The coefficient functions of the considered parabolic equations are periodically oscillating with period ε, where the parameter ε denotes the ratio between the charactersitic microscopic and macroscopic length scales. In addition, the coefficients depend in a possibly discontinuous manner on the macroscopic scale so that real heterogeneities are allowed. In view of greater structural insight and less computational effort, it is our aim to rigorously derive effective equations as ε tends to zero such that solutions of the original model converge to solutions of the effective model. To account for the periodic microstructure as well as for the different diffusion length scales, we employ the method of two-scale convergence via periodic unfolding.

In the first part of the thesis, we consider reaction-diffusion systems, where for some species the diffusion length scale is of order $O(1)$ and for other species it is of order $O(\varepsilon)$. The reaction terms are globally Lipschitz continuous, however, they are in general not the gradient of a given potential. The different diffusivities accompany a loss of compactness such that we cannot pass directly to the limit as ε tends to zero with the nonlinear terms. Based on the notion of strong two-scale convergence, we prove that the effective model is a two-scale reaction-diffusion system depending on the macroscopic and the microscopic scale. In the first step of the proof, we derive Gronwall-type estimates with error terms, and in the second step, we control these errors as ε tends to zero. Our approach supplies explicit rates for the convergence of the solution of the original model to the solution of the effective model.

In the second part, we consider Cahn–Hilliard-type equations with position-dependent mobilities and general potentials. It is well-known that the classical Cahn–Hilliard equation admits a gradient structure which consits of a λ-convex energy functional and a quadratic dissipation potential. Using this gradient structure, we reformulate the parabolic equation via variational inequalities or via the energy-dissipation principle. Based on the Γ-convergence of the energies and the dissipation potentials, we prove evolutionary Γ-convergence, short E-convergence, for the associated gradient systems such that we obtain in the limit as ε tends to zero a Cahn–Hilliard equation with effective (homogenized) coefficients. Moreover, we provide one exemplary potential such that the associated energy functional is not λ-convex and yet we prove E-convergence via the energy-dissipation principle.

Zusammenfassung

Ziel dieser Arbeit ist es zwei verschiedene Klassen von Systemen nichtlinearer parabolischer Gleichungen zu homogenisieren, und zwar Reaktions-Diffusions-Systeme mit verschiedenen Diffusionslängenskalen und Gleichungen vom Typ Cahn–Hilliard. Wir betrachten parabolische Gleichungen mit ε-periodischen Koeffizienten, wobei der Parameter ε das Verhältnis der charakteristischen mikroskopischen und makroskopischen Längenskalen beschreibt. Die Koeffizienten können zusätzlich, möglicherweise unstetig, von der makroskopischen Skale abhängen, sodass echte räumliche Heterogenitäten zugelassen werden können. Unser Ziel ist es, effektive Gleichungen rigoros herzuleiten, um die betrachteten Systeme besser zu verstehen und den Simulationsaufwand zu minimieren. Wir suchen also einen Konvergenzbegriff, mit dem die Lösung des Ausgangsmodells im Limes ε gegen Null gegen die Lösung des effektiven Modells konvergiert. Um die periodische Mikrostruktur und die verschiedenen Diffusivitäten zu erfassen, verwenden wir die Zwei-Skalen Konvergenz mittels periodischer Auffaltung.

Der erste Teil der Arbeit handelt von Reaktions-Diffusions-Systemen, in denen einige Spezies mit der charakteristischen Diffusionslänge $O(1)$ und andere mit $O(\varepsilon)$ diffundieren. Obwohl wir die Reaktionsterme als global Lipschitz-stetig annehmen, sind sie im Allgemeinen nicht der Gradient eines gegebenen Potentials. Die verschiedenen Diffusivitäten führen zu einem Verlust der Kompaktheit, sodass wir nicht direkt den Grenzwert der nichtlinearen Terme bestimmen können. Wir beweisen mittels starker Zwei-Skalen Konvergenz, dass das effektive Modell ein zwei-skaliges Modell ist, welches von der makroskopischen und der mikroskopischen Skale abhängt. Im ersten Schritt des Beweises leiten wir Abschätzungen vom Typ Gronwall mit zusätzlichen Fehlertermen her. Im zweiten Schritt zeigen wir, dass diese Fehler gegen Null konvergieren für ε gegen Null. Unsere Methode erlaubt es uns darüber hinaus, explizite Raten für die Konvergenz der Lösungen des Ausgangsmodells gegen die Lösung des effektiven Modells zu bestimmen.

Im zweiten Teil betrachten wir Gleichungen vom Typ Cahn–Hilliard, welche ortsabhängige Mobilitätskoeffizienten und allgemeine Potentiale beinhalten. Für die klassische Cahn–Hilliard Gleichung ist eine Gradientenstruktur mit einem λ-konvexen Energiefunktional und einem quadratischen Dissipationspotential bekannt. Unter Verwendung dieser Gradientenstruktur, formulieren wir die parabolische Gleichung mittels variationeller Ungleichungen und dem Energie-Dissipations-Prinzip um. Wir beweisen evolutionäre Γ-Konvergenz, kurz E-Konvergenz, der zugehörigen Gradientensysteme basierend auf der Γ-Konvergenz der Energien und der Dissipationspotentiale. Im Limes ε gegen Null erhalten wir eine Cahn–Hilliard Gleichung mit effektiven (homogenisierten) Koeffizienten. Des Weiteren geben wir ein Beispielpotential an, dessen assoziiertes Energiefunktional nicht λ-konvex ist, und beweisen dennoch E-Konvergenz mit dem Energie-Dissipations-Prinzip.

Contents

Introduction

The modeling of many problems in natural sciences requires the consideration of effects on different characteristic length scales. If one scale is negliglibly small compared to another one, it is desirable to derive an *effective* description of these problems in view of greater structural insight and less computational effort.

This thesis is devoted to the development of new mathematical methods to rigorously derive effective equations for *systems of nonlinear parabolic equations*. The problems under consideration comprise nonlinear terms demanding more sophisticated analytical techniques than purely linear systems. We concentrate on problems involving two characteristic length scales, namely the macroscopic and the microscopic scale. The *microstructure* is encoded in the coefficients of the parabolic equations, and we consider coefficients that are essentially *periodic* with respect to the microscopic scale and possibly discontinuous with respect to the macroscopic scale. The presented analytical methods allow us in particular to consider real heterogeneities as shown in Figure 1. We denote with ε the ratio of the microscopic length scale divided by the macroscopic one and we are interested in the regime $\varepsilon \ll 1$.

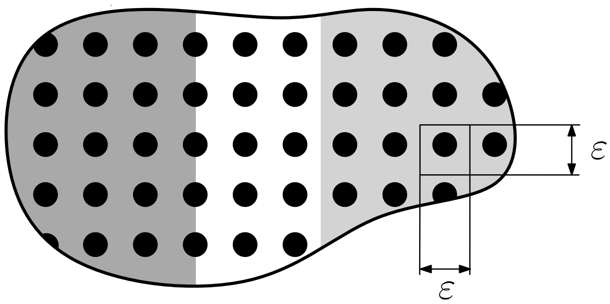

Figure 1: Exemplary coefficient function in two space dimensions: The black dots represent the microstructure and the gray areas stand for the macroscopic heterogeneities.

To derive the effective equations, we employ the concept of two-scale convergence via periodic unfolding. Therefore, we speak of “two-scale homogenization” although our effective models are not always “homogenized” in the classical sense, i.e. the effective model is not necessarily a one-scale model. We consider two types of nonlinear parabolic equations:

1. Two-scale homogenization of reaction-diffusion systems involving different diffusion length scales;

2. Homogenization of Cahn–Hilliard-type equations via evolutionary Γ-convergence.

These two problems indeed belong to two distinct classes of parabolic partial differential equations. While the Cahn–Hilliard equation admits a *gradient structure*, the reaction-diffusion systems under consideration can in general not be formulated as a gradient flow equation. The distinctiveness of both problems requires different analytical methods. To derive the homogenized Cahn-Hilliard equation, we exploit the gradient structure and apply the concept of *evolutionary* Γ*-convergence* based on either energy-dissipation principles or variational inequalities. For general reaction-diffusion systems however, such methods based on gradient structures do not apply. Instead, we work on the level of partial differential equations and derive *Gronwall-type estimates* with controlled *error terms*. These error terms admit quantitative estimates which enable us to provide explicit rates for the convergence of the original problem's solutions to the solution of the effective problem.

1. Two-scale homogenization of reaction-diffusion systems involving different diffusion length scales

In the first part of this thesis, the following reaction-diffusion system is studied: On the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, we consider for $t \geq 0$, $x \in \Omega$

$$\begin{aligned} \partial_t u_\varepsilon(t,x) &= \operatorname{div}(D_1^\varepsilon(x)\nabla u_\varepsilon(t,x)) + F_1^\varepsilon(x,u_\varepsilon(t,x),v_\varepsilon(t,x)), \\ \partial_t v_\varepsilon(t,x) &= \operatorname{div}(\varepsilon^2 D_2^\varepsilon(x)\nabla v_\varepsilon(t,x)) + F_2^\varepsilon(x,u_\varepsilon(t,x),v_\varepsilon(t,x)). \end{aligned} \tag{1.P$_\varepsilon^{\mathrm{RD}}$}$$

No-flux boundary conditions are added on $\partial\Omega$. The variables $u_\varepsilon \in \mathbb{R}^{m_1}$ and $v_\varepsilon \in \mathbb{R}^{m_2}$ denote the concentration vectors of m_1 *classically* and m_2 *slowly* diffusing species for $m_i \in \mathbb{N}$. The term "slowly diffusing" refers to the factor ε^2 in front of the diffusion tensor D_2^ε, while D_1^ε induces classical diffusion, where $D_i^\varepsilon(x) \in \operatorname{Lin}(\mathbb{R}^{m_i\times d};\mathbb{R}^{m_i\times d})$. The coupling of the variables $(u_\varepsilon, v_\varepsilon)$ occurs via the reaction terms $(F_1^\varepsilon, F_2^\varepsilon)$, where $F_i^\varepsilon(x,u_\varepsilon,v_\varepsilon) \in \mathbb{R}^{m_i}$. To focus on the difficulties of passing to the limit $\varepsilon \to 0$ in (1.P$_\varepsilon^{\mathrm{RD}}$), we avoid any questions concerning global existence or positivity of the concentrations by making the simplifying assumption

$$(F_1^\varepsilon, F_2^\varepsilon) \text{ is differentiable and globally Lipschitz continuous in } (u_\varepsilon, v_\varepsilon).$$

However, allowing for nonlinear reaction terms significantly complicates the limit passage $\varepsilon \to 0$ in (1.P$_\varepsilon^{\mathrm{RD}}$) as we explain below; in particular, since $(F_1^\varepsilon, F_2^\varepsilon)$ is not a gradient. The underlying microstructure of the system is encoded in the given data as visualized in Figure 1 and we assume that the data, cf. (2.1.14.Conv),

$$D_i^\varepsilon(x) \text{ and } F_i^\varepsilon(x,u,v) \text{ converge in the two-scale sense to } \mathbb{D}_i(x,y) \text{ and } \mathbb{F}_i(x,y,u,v).$$

The two-scale functions $\mathbb{D}_i$ and $\mathbb{F}_i$ are periodic in the y-component with respect to the so-called *periodicity cell* $\mathcal{Y} = \mathbb{R}^d/_{\mathbb{Z}^d}$, which is obtained from the unit cell $Y = [0,1)^d$ by identifying opposite faces. Our analysis covers of course the choice $D_i^\varepsilon(x) = \mathbb{D}_i(x, x/\varepsilon)$ and $F_i^\varepsilon(x,u,v) = \mathbb{F}_i(x, x/\varepsilon, u, v)$.

In Chapter 2, we show that for $\varepsilon \to 0$, the limit model is a two-scale model given for $t \geq 0$ on $(x,y) \in \Omega \times \mathcal{Y}$ by

$$\begin{aligned} \partial_t u(t,x) &= \operatorname{div}(D_{\mathrm{eff}}(x)\nabla u(t,x)) + F_{\mathrm{eff}}(x,u(t,x),V(t,x,*)), \\ \partial_t V(t,x,y) &= \operatorname{div}_y(\mathbb{D}_2(x,y)\nabla_y V(t,x,y)) + \mathbb{F}_2(x,y,u(t,x),V(t,x,y)). \end{aligned} \tag{2.P$_0^{\mathrm{RD}}$}$$

While the effective diffusion tensor $D_{\text{eff}}(x)$ is given for every $x \in \Omega$ via the standard unit cell problem, cf. (2.1.16), the reaction term F_{eff} is the average

$$F_{\text{eff}}(x, u(t,x), V(t,x,*)) = \fint_{\mathcal{Y}} \mathbb{F}_1(x, y, u(t,x), V(t,x,y))\,\mathrm{d}y,$$

but depending on the microscopic function $V(t,x,*)$. In contrast, the effective data $\mathbb{D}_2$ and $\mathbb{F}_2$ in $(2.\mathrm{P}_0^{\mathrm{RD}})_2$ are indeed two-scale functions, i.e. they additionally depend on $y \in \mathcal{Y}$. While the V-equations contain no derivatives with respect to the macroscopic scale $x \in \Omega$, they rather define parabolic partial differential equations on the periodicity cell $\mathcal{Y}$. With this, the effective equations in $(2.\mathrm{P}_0^{\mathrm{RD}})_2$ cannot be reduced to a one-scale model as in $(2.\mathrm{P}_0^{\mathrm{RD}})_1$, which resembles the structure of the original model in $(1.\mathrm{P}_\varepsilon^{\mathrm{RD}})_1$.

There are many mathematical publications on two-scale homogenization of reaction-diffusion systems involving different diffusion length scales which often aim at applications in biology, chemistry, and engineering. For a review of mathematical publications on periodic homogenization for parabolic partial differential equations, we refer to Subsection 2.1.1. The derivation of effective coefficients for systems of reaction-diffusion type in heterogeneous media is also of great interest in various physical applications, see e.g. [BeK83, Xin00, Kee00, ABK09].

Analytical difficulties arising with slow diffusion

The difficulty arising with system $(1.\mathrm{P}_\varepsilon^{\mathrm{RD}})$ is the degeneracy of the $\mathrm{H}^1(\Omega)$-norm for v_ε, namely v_ε is only bounded pointwise in time via

$$\sup_{\varepsilon>0} \Big\{ \|v_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \varepsilon \|\nabla v_\varepsilon\|_{\mathrm{L}^2(\Omega)} \Big\} < \infty. \tag{3}$$

To deal with the underlying periodic microstructure as well as with the degeneracy of the v_ε-norm in (3), we employ the method of two-scale convergence via periodic unfolding. Therefore, we introduce the *periodic unfolding operator* $\mathcal{T}_\varepsilon : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ following [CDG02]: Defining $\mathcal{T}_\varepsilon\, u(x,y) := u_{\text{ex}}(\varepsilon[x/\varepsilon] + \varepsilon y)$, where $[\,\cdot\,] : \mathbb{R}^d \to \mathbb{Z}^d$ maps every point in $\mathbb{R}^d$ to its nearest lattice point in $\mathbb{Z}^d$, the periodic unfolding operator maps one-scale functions to two-scale functions. Here, $u_{\text{ex}} \in \mathrm{L}^2(\mathbb{R}^d)$ is obtained from $u \in \mathrm{L}^2(\Omega)$ by extension with 0 outside of Ω. With the aid of $\mathcal{T}_\varepsilon$, *weak and strong two-scale convergence* of $(u_\varepsilon)_\varepsilon$ is defined via classical weak and strong convergence of $(\mathcal{T}_\varepsilon\, u_\varepsilon)_\varepsilon$ in the two-scale space $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$.

Based on periodic unfolding, we have the following *compactness result*, cf. Theorem 1.2.5: If $(v_\varepsilon)_\varepsilon \subset \mathrm{H}^1(\Omega)$ satisfies the a priori bound (3), then there exists a two-scale function $V \in \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y}))$ and we have weak two-scale convergence (up to subsequences) in the following sense

$$\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup V_{\text{ex}} \quad \text{and} \quad \mathcal{T}_\varepsilon(\varepsilon \nabla v_\varepsilon) \rightharpoonup \nabla_y V_{\text{ex}} \quad \textit{weakly} \text{ in } \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y}). \tag{4}$$

We point out that $\mathrm{H}^1(\mathcal{Y}) \subset \mathrm{H}^1(Y)$ is the closed subspace with periodic boundary values. The crucial observation is that slow diffusion accompanies a loss of compactness, namely $\mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y})) \subset \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ continuously, but not compactly. Therefore, (4) does *not* imply $\mathcal{T}_\varepsilon(v_\varepsilon) \to V^{\text{ex}}$ *strongly* in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ so that we cannot pass to the limit $\varepsilon \to 0$ with nonlinear reaction terms. However, using the a priori weak convergence in (4), one can derive rigorous homogenization results for linear systems. Moreover, methods of convex analysis apply to reaction terms which are gradients of convex potentials and the formal method of asymptotic expansion is suited for truly nonlinear systems – as we discuss in more detail in Section 2.1.

We emphasize that for the classically diffusing variable u_ε we have a priori $u_\varepsilon \to u$ strongly in $\mathrm{L}^2(\Omega)$ (up to subsequences) thanks to the compact embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$. To rigorously derive effective equations for the coupled system (1.$\mathrm{P}_\varepsilon^{\mathrm{RD}}$), we focus on proving strong two-scale convergence for the slowly diffusing variable v_ε. Therefore, our approach is designed for v_ε, however, it also applies to u_ε.

Our approach to slow diffusion and nonlinear reaction

For v_ε and V denoting the weak solutions of (1.$\mathrm{P}_\varepsilon^{\mathrm{RD}}$)$_2$ and (2.$\mathrm{P}_0^{\mathrm{RD}}$)$_2$, respectively, the first main result of this thesis is the rigorous proof of

$$\| \mathcal{T}_\varepsilon v_\varepsilon(t) - V^{\mathrm{ex}}(t)\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} \xrightarrow{\varepsilon\to 0} 0 \quad \text{uniformly in } [0,T]. \tag{5}$$

We neither assume that v_ε admits an asymptotic expansion in ε nor that V is continuous with respect to $x \in \Omega$ or $y \in \mathcal{Y}$. Note that, if V were spatially continuous, then (5) would be equivalent to $\|v_\varepsilon(t) - [V]^\varepsilon(t)\|_{\mathrm{L}^2(\Omega)} \xrightarrow{\varepsilon\to 0} 0$, where $[V]^\varepsilon(x) := V(x, x/\varepsilon)$.

To prove (5), we are following a similar approach as in [Eck05] based on Gronwall-type estimates and refer to Section 2.1.4 for an outline of the general strategy. In the first step we derive an estimate of the form

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\| \mathcal{T}_\varepsilon v_\varepsilon(t) - V^{\mathrm{ex}}(t)\|_2^2 \le L\| \mathcal{T}_\varepsilon v_\varepsilon(t) - V^{\mathrm{ex}}(t)\|_2^2 + \Delta_1^\varepsilon(t) + \Delta_2^\varepsilon(t) + \Delta_3^\varepsilon(t) + \Delta_4^\varepsilon(t),$$

where L denotes the global Lipschitz constant of the reaction terms. And in the second step, we show that the error terms Δ_i^ε converge pointwise to 0. To prove the latter estimate, we reformulate the weak formulation of (1.$\mathrm{P}_\varepsilon^{\mathrm{RD}}$)$_2$ via periodic unfolding and obtain the *folding mismatch* Δ_1^ε due to different regularity properties of the folding operator $\mathcal{F}_\varepsilon : \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y}) \to \mathrm{L}^2(\Omega)$ and the gradient folding operator $\mathcal{G}_\varepsilon^1 : \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y})) \to \mathrm{H}^1(\Omega)$, cf. Section 1.2.4. Owing to the fact that unfolded functions are not Y-periodic in general, namely $\mathcal{T}_\varepsilon v_\varepsilon \in \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1(Y)) \supsetneq \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1(\mathcal{Y}))$ for all $v_\varepsilon \in \mathrm{H}^1(\Omega)$, the *periodicity defect error* Δ_2^ε arises. However, if the two-scale limit $U^{\mathrm{ex}} = \text{w-}\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon u_\varepsilon$ exists, then it is indeed Y-periodic. We call this effect $\mathcal{T}_\varepsilon$*-property of recovered periodicity.* The error terms Δ_3^ε and Δ_4^ε capture the standard *approximation errors* for the given data D_i^ε and F_i^ε, respectively.

As a further technical issue let us mention that a priori $\partial_t v_\varepsilon(t) \in \mathrm{H}^1(\Omega)^*$, merely, whereas the operator $\mathcal{T}_\varepsilon$ is well-defined for integrable functions, only. We therefore improve the time-regularity of weak solutions by imposing differentiability of (F_1, F_2) and by considering initial values $(u_\varepsilon(0), v_\varepsilon(0))$ and $(u(0), V(0))$ with additional regularity, cf. Proposition 1.1.3. We point out that such additional assumptions are comparable to well-prepared initial conditions within the context of evolutionary Γ-convergence.

With this, we arrive in Section 2.1 at (cf. Theorem 2.1.1)

Main Theorem I: (5) holds for solutions with improved time-regularity.

The article [MRT14] provides the basis for Main Theorem I in Section 2.1 as well as for Chapter 1 on preliminary results concerning the existence of solutions and two-scale convergence.

In Section 2.2 we are able to show that solutions with improved time-regularity can be approximated by solutions without improved time-regularity such that this approximation is compatible with the homogenization limit. Therefore, the additional assumption on the initial values becomes dispensable and we obtain (cf. Theorem 2.2.1)

Main Theorem II: (5) holds for solutions without improved time-regularity.

To include the classically diffusing variable u_ε is straight forward so that we can pass to the limit $\varepsilon \to 0$ with the whole system in $(1.\mathrm{P}_\varepsilon^{\mathrm{RD}})$. Our approach is in particular strong enough to supply quantitative estimates.

Quantitative estimates

Up to the present, we have put the emphasis on the convergence of the slowly diffusing variable v_ε, since the limit passage $\varepsilon \to 0$ in $(1.\mathrm{P}_\varepsilon^{\mathrm{RD}})$ has been well-known for the classically diffusing variable u_ε. However, to quantify their convergence in Section 2.3, we consider u_ε and v_ε simultaneously. Assuming higher regularity for the given data and the effective solution (u, V) of $(2.\mathrm{P}_0^{\mathrm{RD}})$ with respect to $x \in \Omega$, we prove quantitative estimates for the error terms Δ_i^ε. With this, we can provide explicit rates for the convergence of the solutions

Main Theorem IIIa–b:

$$\| \mathcal{T}_\varepsilon v_\varepsilon(t) - V(t)\|_{C([0,T];\mathrm{L}^2(\Omega\times\mathcal{Y}))} + \|u_\varepsilon(t) - u(t)\|_{C([0,T];\mathrm{L}^2(\Omega))} \leq C\varepsilon^\eta,$$

where $\eta > 0$ depends on the choice of the initial values, cf. Theorem 2.3.1 and 2.3.2. Theorem 2.3.1, which proves $\eta = \frac{1}{4}$ and which is the main result of Section 2.3, is published in the preprint [Rei14]. We point out that we do not assume higher regularity for the original solutions $(u_\varepsilon, v_\varepsilon)$ or for the corrector functions.

Based on the explicit definitions of the periodic unfolding and folding operators $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$, respectively, we obtain quantitative estimates for the approximation errors Δ_3^ε and Δ_4^ε in a straight forward way. To estimate the periodicity defect Δ_2^ε, we rely on results in [Gri04, Gri05]. For the estimation of the folding mismatch Δ_1^ε, we use the scale-splitting operator $\mathcal{Q}_\varepsilon$ introduced in [CDG02]. The operator $\mathcal{Q}_\varepsilon$ allows us to prove quantitative estimates for the difference between $\mathcal{F}_\varepsilon U$ and $\mathcal{G}_\varepsilon^\gamma U$ for $\gamma \in \{0, 1\}$ quite easily in the case that $U(x, y) = w(x)z(y)$ is a tensor product. While in the case of exact periodicity and classical diffusion, the corrector satisfies $U(x, y) = \sum_{j=1}^d \frac{\partial u}{\partial x_j}(x) \cdot z_j(y)$, cf. Section 2.1.3, the two-scale limit $V(x, y)$ is not a product in general. We overcome this difficulty in Proposition 2.3.8 by using an orthogonality argument: For an orthonormal basis $\{\Phi_j\}_j$ in $\mathrm{H}^1(\mathcal{Y})$, the family $\{\mathcal{G}_\varepsilon^\gamma \Phi_j\}_j$ is orthogonal in $\mathrm{H}^1(\Omega)$.

We emphasize that our quantitative estimates apply particularly to linear elliptic systems with degeneracy of slow diffusion type (cf. Proposition 2.3.17) and our estimates provide new results even in this simpler case. For a comparison of our convergence rates with others, we refer to Subsection 2.3.9.

2. Homogenization of Cahn–Hilliard-type equations via evolutionary Γ-convergence

In the second part of this thesis we study Cahn–Hilliard-type equations modeling the phase separation of two different species. For $t \geq 0$ and $x \in \Omega$ this is done by

$$\partial_t u_\varepsilon(t,x) = \operatorname{div}\left[M_\varepsilon(x)\nabla\big(\partial_u W_\varepsilon(x,u_\varepsilon(t,x)) - \operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon(t,x))\big)\right], \tag{6.$\mathrm{P}_\varepsilon^{\mathrm{CH}}$}$$

where the variable u_ε is scalar-valued and describes the spatial distribution of the two species. For instance, u_ε takes the value -1 if only species one is present and u_ε is equal to $+1$ if only species two is present. We supplement the fourth-order equation (6.$\mathrm{P}_\varepsilon^{\mathrm{CH}}$) with no-flux boundary conditions for the concentration u_ε and the chemical potential $\partial_u W_\varepsilon(x,u_\varepsilon(t,x)) - \operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon(t,x))$. The equation's dynamics are determined by the mobility tensor $M_\varepsilon(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, the tensor $A_\varepsilon(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ which is related to the interfacial transition layer and the potential $W_\varepsilon(x,u_\varepsilon) \in [0,\infty]$, which enforces the separation of the two species. As before, the (not exactly) periodically oscillating coefficients capture the underlying microstructure. For a physical application, we refer to [BK*02, TB*03], where the dewetting process of thin films on heterogeneous substrates is studied.

In Chapter 3, which is based on [LiR15], we show that solutions $(u_\varepsilon)_\varepsilon$ of (6.$\mathrm{P}_\varepsilon^{\mathrm{CH}}$) converge for $\varepsilon \to 0$ to the solution u of the effective equation

$$\partial_t u(t,x) = \operatorname{div}\left[M_{\mathrm{eff}}(x)\nabla\big(\partial_u W_{\mathrm{eff}}(x,u(t,x)) - \operatorname{div}(A_{\mathrm{eff}}(x)\nabla u(t,x))\big)\right]. \tag{7.$\mathrm{P}_0^{\mathrm{CH}}$}$$

Similar to the u-equations in (2.$\mathrm{P}_0^{\mathrm{RD}}$)$_1$, the effective tensors $M_{\mathrm{eff}}(x)$ and $A_{\mathrm{eff}}(x)$ are determined for all $x \in \Omega$ by the unit cell problem, cf. (3.2.16) and (3.2.18). In the case of $W_\varepsilon(x,u) = \mathbb{W}(x,x/\varepsilon,u)$, the effective potential W_{eff} is given by averaging $\mathbb{W}$ with respect to the periodicity cell $\mathcal{Y}$. We emphasize that the effective equation (7.$\mathrm{P}_0^{\mathrm{CH}}$) is of the same structure as the original one (6.$\mathrm{P}_\varepsilon^{\mathrm{CH}}$), namely a parabolic equation in the macroscopic domain Ω.

Evolutionary Γ-convergence of gradient systems

We begin by studying evolutionary Γ-convergence of abstract gradient systems $(X,\mathcal{E},\mathcal{R})$. The space X denotes the set of admissible states and the functionals $\mathcal{E}$ and $\mathcal{R}$ are the energy or entropy and the dissipation of the system, respectively. We consider only "classical" gradient systems meaning that the dissipation potential $\mathcal{R}(\dot u) = \frac{1}{2}\langle \dot u, \mathbb{G}\dot u\rangle$ is a quadratic functional. The evolution of the gradient system is determined by the abstract balance between viscous and potential restoring forces which can formally be written as the *force-balance formulation*

$$0 = \mathrm{D}\mathcal{E}(u(t)) + \mathrm{D}\mathcal{R}(\dot u(t)). \tag{8}$$

The precise notion of the differentials (or possibly set-valued subdifferentials) $\mathrm{D}\mathcal{E}$ and $\mathrm{D}\mathcal{R}$ is postponed to Section 3.1. Two major advantages of gradient systems are: (a) They reflect the physical principle of energy minimization, and (b) many mathematical tools based on variational methods apply.

Considering now families of functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, our aim is to broaden the understanding and applicability of mathematical methods to derive limit functionals $\mathcal{E}_0$ and $\mathcal{R}_0$ as $\varepsilon \to 0$. The term 'evolutionary Γ-convergence', short *E-convergence*, is related to the fact that static Γ-convergence results can often be shown for each functional $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$. To guarantee that solutions $u_\varepsilon : [0,T] \to X$ of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converge to solutions of $(X, \mathcal{E}_0, \mathcal{R}_0)$, the compatibility of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ is essential.

We consider two approaches to E-convergence: The first approach relies on the uniform Λ-convexity of $\mathcal{E}_\varepsilon$ with respect to $\mathcal{R}_\varepsilon$ and it is based on [AGS05, DaS08, DaS10, Mie14]. For such Λ-convex gradient systems the force-balance formulation (8) is equivalent to the following *Integrated Evolutionary Variational Estimate (IEVE)*:

$$\begin{aligned}&\text{For all } 0 \le s < t \text{ and all } w \in \operatorname{dom}(\mathcal{E}_\varepsilon) :\\ &\mathrm{e}^{\Lambda(t-s)} \left(\mathcal{R}_\varepsilon(u_\varepsilon(t) - w) - \mathcal{R}_\varepsilon(u_\varepsilon(s) - w)\right) \le \mathrm{M}_\Lambda(t-s) \left(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon(t))\right),\end{aligned}$$

where $\mathrm{M}_\Lambda(r) = \int_0^r \mathrm{e}^{\Lambda\tau}\,\mathrm{d}\tau$. Under the assumptions that $\mathcal{E}_\varepsilon$ strongly Γ-converges to $\mathcal{E}_0$ and that $\mathcal{R}_\varepsilon$ continuously converges to $\mathcal{R}_0$, we can pass to the limit $\varepsilon \to 0$ in IEVE. This is formulated as our first abstract E-convergence result in Theorem 3.1.5.

The second approach to E-convergence uses the Legendre–Fenchel transform $\mathcal{R}^*_\varepsilon$ of $\mathcal{R}_\varepsilon$. Under the assumption that $\mathcal{E}_\varepsilon$ satisfies a chain rule condition, the force-balance formulation (8) is equivalent to the *Energy-Dissipation Principle (EDP)*

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\,\mathrm{d}t \le \mathcal{E}_\varepsilon(u_\varepsilon(0)).$$

For this formulation, the energies $\mathcal{E}_\varepsilon$ do not need to satisfy any convexity condition. In Theorem 3.1.6, we prove the second abstract result: $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges to $(X, \mathcal{E}_0, \mathcal{R}_0)$, if the following three conditions hold: (i) The functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ strongly Γ-converge to $\mathcal{E}_0$ and $\mathcal{R}_0$, respectively, (ii) the energy subdifferentials are closed, cf. (3.1.23), and (iii) the initial conditions are *well-prepared*, namely $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$. Theorem 3.1.6 is a new result which generalizes [Mie14, Thm. 3.6] with respect to the assumptions on $\mathcal{R}_\varepsilon$ (and we refer to Section 3.3 for more details). The EDP approach is based on the well-known Sandier–Serfaty principle [SaS04]. However, therein the setting is formulated in a very general manner so that the verification for a special application is very difficult. In contrast, we prove E-convergence under explicit conditions for $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, and in particular, we do not impose two separate estimates for the primal and dual dissipation potentials.

Homogenization of Cahn–Hilliard-type equations

To homogenize the Cahn–Hilliard-type equation ($6.\mathrm{P}^{\mathrm{CH}}_\varepsilon$) with spatially oscillating coefficients, we apply the theory of evolutionary Γ-convergence for gradient systems. For this purpose, we have to identify the triple $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ and check the assumptions for E-convergence in the IEVE formulation (Theorem 3.2.10) respectively in the EDP formulation (Theorem 3.2.13). On the state space X, which is the dual of H^1-functions with fixed average, the energy functional and the dissipation potential are given via

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{1}{2}\nabla u \cdot A_\varepsilon(x)\nabla u + W_\varepsilon(x,u)\,\mathrm{d}x \quad \text{and} \quad \mathcal{R}_\varepsilon(\dot u) = \int_\Omega \frac{1}{2}\nabla \xi_{\dot u} \cdot M_\varepsilon(x)\nabla \xi_{\dot u}\,\mathrm{d}x,$$

where $\xi_{\dot{u}}$ is the solution of $-\operatorname{div}(M_\varepsilon(x)\nabla\xi_{\dot{u}}) = \dot{u}$. Assuming the potentials W_ε to be uniformly continuous with respect to u and satisfying suitable growth conditions, we obtain the Γ-convergence for the energies.

The convexity of $\mathcal{E}_\varepsilon$ depends on the choice the potential W_ε. If the potential W_ε is λ-convex with respect to u, then $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ is convex and the gradient system has many useful properties. For instance, the Fréchet subdifferential $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon$ exists, the chain rule is satisfied, and the closedness condition (3.1.23) holds. In this case, both approaches, the IEVE and the EDP formulation, apply and the E-convergence of the system $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ follows so that we obtain the effective Cahn–Hilliard-type equation ($7.\mathrm{P}_0^{\mathrm{CH}}$). Exemplary potentials fitting into the Λ-convex setting are the well-known double-well and logarithmic potentials. Moreover, we provide a non λ-convex potential, which is a counterexample for the IEVE formulation, and we verify the conditions (i)–(iii) so that E-convergence via the EDP formulation also holds in this case.

We emphasize that the application of E-convergence via IEVE or EDP is not restricted to equations, but applies to systems of equations as well. For a review of existing literature on E-convergence and homogenization results related to the Cahn–Hilliard equation, we refer to Subsection 3.2.1.

Structure of the thesis

The first part of this thesis comprises Chapter 1 *Preliminaries* and Chapter 2 *Reaction-diffusion systems involving different diffusion length scales*. Chapter 1 contains two sections, whereas the first one is devoted to general systems of reaction-diffusion type with Lipschitz continuous nonlinearities and the second one is devoted to the theory of two-scale convergence. In Chapter 2, we consider the asymptotic behavior of the coupled system ($1.\mathrm{P}_\varepsilon^{\mathrm{RD}}$) and we prove Main Theorem I, II and IIIa–b in Section 2.1, 2.2 and 2.3, respectively.

The second part of this thesis, Chapter 3, deals with *Homogenization of Cahn-Hilliard-type equations via evolutionary* Γ*-convergence*, which consists of mainly two sections. The first section is devoted to E-convergence of abstract gradient systems and the second one contains the homogenization result for the Cahn-Hilliard equation ($6.\mathrm{P}_\varepsilon^{\mathrm{CH}}$).

1 Preliminaries

1.1 General reaction-diffusion systems

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $T > 0$ be fixed. The focus of this section and Chapter 2 are semilinear reaction-diffusion equations of the type

$$\begin{aligned} u_t &= \mathcal{A}u + F(u) && \text{in } [0,T] \times \Omega, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{1.1.1.P}$$

Here $\mathcal{A}$ denotes an elliptic differential operator of the form $\mathcal{A}(t,x)u := \operatorname{div}(D(t,x)\nabla u)$ supplied with homogeneous Neumann boundary conditions, i.e. $(D\nabla u) \cdot \nu = 0$ on $[0,T] \times \partial\Omega$. Here, $\nu \in \mathbb{R}^d$ denotes the unit outer normal vector of Ω and $(D\nabla u) \cdot \nu$ is a vector in $\mathbb{R}^m$. We abbreviate the partial time derivative $\partial_t u$ with u_t. For the application we have in mind, $u : [0,T] \times \Omega \to \mathbb{R}^m$ denotes the concentration, $D : [0,T] \times \Omega \to \mathbb{R}^{(m\times d)\times(m\times d)}$ the diffusion tensor, and $F : [0,T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ the reaction term.

Both systems, $(1.\mathrm{P}^{\mathrm{RD}}_{\varepsilon})$ and $(2.\mathrm{P}^{\mathrm{RD}}_{0})$, can be reformulated in terms of (1.1.1.P). In this section, we present a mathematical setting that accounts for both systems and that is independent of $\varepsilon > 0$ and $y \in \mathcal{Y}$.

Section 1.1 is structured as follows. We introduce the notion of solutions in Subsection 1.1.1 and give results concerning the existence of solutions and improved time-regularity in Subsection 1.1.2. In Subsection 1.1.3, we discuss the assumptions and present an exemplary reaction-diffusion system.

1.1.1 Notion of solution and data qualification

Let X and H denote two Hilbert spaces. We denote with X^* the dual space of X and with $\langle \cdot, \cdot \rangle_{X^*,X}$ the associated dual pairing. We assume that H can be identified with its dual, i.e. $H = H^*$, and we write $(\cdot,\cdot)_H$ for a scalar product on H. Assume that X is dense and continuously embedded in H, then we obtain the evolution triple $X \subset H \subset X^*$. If not indicated otherwise, we set for $m \in \mathbb{N}$

$$X := \mathrm{H}^1(\Omega;\mathbb{R}^m) \quad \text{and} \quad H := \mathrm{L}^2(\Omega;\mathbb{R}^m) \tag{1.1.2}$$

and call X the *space of test functions*. For $u \in X$, we then have $\nabla u \in \mathrm{L}^2(\Omega;\mathbb{R}^{m\times d})$. The associated Sobolev norms for vector-valued functions are $\|u\|_H = (\sum_{i=1}^m \|u_i\|^2_{\mathrm{L}^2(\Omega)})^{1/2}$ and $\|u\|_X = (\sum_{i=1}^m \sum_{|\alpha|\le 1} \|\partial_x^\alpha u_i\|^2_{\mathrm{H}^1(\Omega)})^{1/2}$. Here, $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_d$ its length and

$$\partial_x^\alpha u_i = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} = \partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}\, u_i.$$

Here and in Chapter 2, we abbreviate the spaces $\mathrm{H}^1(\Omega;\mathbb{R}^m)$ and $\mathrm{L}^2(\Omega;\mathbb{R}^m)$ with $\mathrm{H}^1(\Omega)$ and $\mathrm{L}^2(\Omega)$, respectively.

For the evolution triple $X \subset H \subset X^*$, the relevant space for our analysis is

$$W(0,T;X) := \mathrm{L}^2(0,T;X) \cap \mathrm{H}^1(0,T;X^*). \tag{1.1.3}$$

The continuous embedding of $W(0,T;X)$ into $C([0,T];H)$ holds according to [GGZ74, Thm. 1.17].

Definition 1.1.1. *We call $u \in W(0,T;X)$ a solution of* (1.1.1.P)*, if u satisfies a.e. in $[0,T]$ the weak formulation*

$$\langle u_t, \varphi\rangle_{X^*,X} = (-D\nabla u, \nabla\varphi)_H + \langle F(u), \varphi\rangle_{X^*,X} \quad \textit{for all } \varphi \in X \tag{1.1.4.WF}$$

and it holds $u(0) = u_0$.

Since we are, among others, interested in the homogenization of the reaction term F, we do not want to understand $F(\cdot,\cdot,u(\cdot,\cdot))$ as general distribution (which is sufficient for the existence of solutions), but as an integrable function. Thus, we assume the reaction $F : u \mapsto F(u)$ to be differentiable and globally Lipschitz continuous (and not just locally) which is not too restrictive in practice as Example 2.1 shows. More precisely, the following assumptions hold for the given data.

Uniform Ellipticity: The diffusion tensor $D : [0,T] \times \Omega \to \mathbb{R}^{(m\times d)\times(m\times d)}$ is measurable on Ω and continuously differentiable on $(0,T)$, i.e. $t \mapsto D(t,x) \in C^1(0,T)$ for a.a. $x \in \Omega$. Moreover, D is uniformly elliptic and bounded, namely

$$\exists\, 0 < \alpha \le \beta < \infty : \quad D(t,x)\xi : \xi \ge \alpha|\xi|^2 \quad \text{and} \quad |D(t,x)\xi| \le \beta|\xi| \tag{1.1.5}$$

for all $\xi \in \mathbb{R}^{m\times d}$ and $(t,x) \in [0,T] \times \Omega$.

Lipschitz continuity: The reaction $F : [0,T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function which is differentiable, i.e. $x \mapsto F(t,x,A)$ is measurable for all $(t,A) \in [0,T] \times \mathbb{R}^m$ and $(t,A) \mapsto F(t,x,A) \in C^1((0,T) \times \mathbb{R}^m)$ for a.a. $x \in \Omega$. Moreover, F is globally Lipschitz continuous with respect to A and bounded with respect to x, i.e.

$$\begin{aligned} \exists\, L, C_\infty \ge 0 : \qquad & |F(t,x,A) - F(t,x,B)| \le L|A - B|, \\ & |F(t,x,0)| \le C_\infty, \end{aligned} \tag{1.1.6}$$

for all $A, B \in \mathbb{R}^m$ and $(t,x) \in [0,T] \times \Omega$.

Here $A : B = \mathrm{tr}(A^t B)$ and $\vec{a} \cdot \vec{b}$ denote the scalar product for matrices in $\mathbb{R}^{m\times d}$ and for vectors in $\mathbb{R}^m$, respectively; $|\cdot|$ denotes the induced (matrix respective vector) norm. For the sets of parameters (α,β) and (L, C_∞) with $\alpha,\beta > 0$ and $L, C_\infty \ge 0$, we introduce the *function classes*

$$\begin{aligned} \mathcal{M}(\Omega) &:= \{D : [0,T] \times \Omega \to \mathbb{R}^{(m\times d)\times(m\times d)} \mid D \text{ satisfies (1.1.5) with } (\alpha,\beta)\}, \\ \mathcal{F}(\Omega) &:= \{F : [0,T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}^m \mid F \text{ satisfies (1.1.6) with } (L, C_\infty)\}. \end{aligned} \tag{1.1.7}$$

For our analysis it is not necessary that the tensor D satisfies any symmetry relations. The global Lipschitz continuity of F guarantees $(t,x) \mapsto F(t,x,u(t,x)) \in \mathrm{L}^2(0,T;H)$ for all $u \in \mathrm{L}^2(0,T;H)$. Indeed, using (1.1.6) with $B = 0$ yields the *growth condition*

$$|F(t,x,A)| \leq C_1(1+|A|) \tag{1.1.8}$$

for all $A \in \mathbb{R}^m$ and $(t,x) \in [0,T] \times \Omega$, where $C_1 := \max\{L, C_\infty\}$. With this, we can associate to F a continuous Nemytskii operator $\mathrm{L}^2(0,T;H) \to \mathrm{L}^2(0,T;H)$.

We point out that we do not employ any compactness properties, such as $X \subset H$ compactly, in this section since this does no longer hold true in the two-scale setting in Chapter 2. Indeed, the two-scale space $\mathbb{X} = \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y}))$ is densely and continuously embedded into $\mathbb{H} = \mathrm{L}^2(\Omega \times \mathcal{Y})$, but this embedding is clearly not compact. In the same manner, we also do not apply any Sobolev embeddings of the type $\mathrm{H}^1(\Omega) \subset \mathrm{L}^p(\Omega)$ with $p > 2$, because $\mathbb{X} \not\subseteq \mathrm{L}^p(\Omega \times \mathcal{Y})$ fails again.

The existence result (Theorem 1.1.2) and the homogenization results in Chapter 2 do not rely on the homogeneous Neumann boundary conditions in (1.1.1.P). In the case of non-homogeneous Neumann boundary conditions g, the boundary integral $\int_{\partial\Omega} g \cdot \varphi \,\mathrm{d}\sigma$ would appear as linear term on the right-hand side in (1.1.4.WF). Other choices such as Dirichlet or periodic boundary conditions are admissible as well and then $X = \mathrm{H}^1_0(\Omega)$ or $X = \mathrm{H}^1_{\mathrm{per}}(\Omega)$, respectively, and (1.1.4.WF) holds as it is. A Poincaré-type inequality is not needed.

1.1.2 Existence of solutions and improved time-regularity

We prove the existence of a unique solution for problem (1.1.1.P) by applying Banach's fixed-point theorem. Similar existence results can be found in e.g. [Paz83, Thm. 1.2 p. 184] and [Hen81, Thm. 3.3.3].

Theorem 1.1.2. *Assume that $D \in \mathcal{M}(\Omega)$, $F \in \mathcal{F}(\Omega)$ and $u_0 \in H$. The semilinear problem, which is a generalization of* (1.1.1.P) *via the linear term $f \in \mathrm{L}^2(0,T;X^*)$,*

$$\begin{aligned} u_t &= \mathcal{A}u + F(u) + f \quad && \text{in } [0,T] \times \Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{1.1.9}$$

possesses for every given $T > 0$ a unique solution $u \in W(0,T;X)$. Moreover, there exists a constant $C_a \geq 0$ such that it holds

$$\|u\|_{C([0,T];H)} + \sqrt{\alpha}\|\nabla u\|_{\mathrm{L}^2(0,T;H)} + \|u_t\|_{\mathrm{L}^2(0,T;X^*)} \leq C_a\left(1 + \tfrac{\beta}{\sqrt{\alpha}}\right), \tag{1.1.10}$$

where C_a depends on the given quantities $\|u_0\|_H, T, L, C_\infty$, the domain Ω, and the ratio $\|f\|_{\mathrm{L}^2(0,T;X^*)}/\sqrt{\alpha}$.

Proof. The proof consists of four steps: The Steps 1–2 are devoted to the existence of a unique solution on $[0,T]$ following the lines of [Eva98, ff. 499] or [Smo94, ff. 114]. In the Steps 3–4, we prove the upper bound (1.1.10).

Step 1: Existence of local solutions. We set $V := C([0,T];H)$ with

$$\|u\|_V := \max_{0 \leq t \leq T} \|u(t)\|_H$$

and define the mapping $\Phi : V \to V$ by $\Phi(u) = w$, where $w \in W(0,T;X)$ is the unique solution to the linear initial-boundary value problem

$$\begin{aligned} w_t &= \mathcal{A}w - w + F(u) + u + f \quad \text{ in } [0,T] \times \Omega \\ w(0) &= u_0. \end{aligned}$$

The existence of the solution w follows by Galerkin approximation, see e.g. [Tem88, Thm. 3.4], [Emm04, Thm. 8.4.1], or time-discretization, see e.g. [Emm04, Thm. 8.3.5], and references therein.

We show that $\Phi : V \to V$ is a contraction, by setting $w = \Phi(u)$, $\tilde{w} = \Phi(\tilde{u})$ and $h = F(u)$, $\tilde{h} = F(\tilde{u})$. According to [Emm04, Cor. 8.1.10], it holds $\frac{\mathrm{d}}{\mathrm{d}t}(u,u)_H = 2\langle u, u_t\rangle_{X,X^*}$ a.e. in $[0,T]$. Then, partial integration and the ellipticity (1.1.5) yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}&\|w-\tilde{w}\|_H^2 + 2\alpha\|\nabla w - \nabla\tilde{w}\|_H^2 \\ &\leq \frac{\mathrm{d}}{\mathrm{d}t}\|w-\tilde{w}\|_H^2 + 2(D(\nabla w - \nabla\tilde{w}), \nabla w - \nabla\tilde{w})_H \\ &= 2\langle w-\tilde{w}, w_t - \tilde{w}_t\rangle_{X,X^*} - 2\langle w-\tilde{w}, \operatorname{div}(D\nabla w) - \operatorname{div}(D\nabla\tilde{w})\rangle_{X,X^*} \\ &= 2\langle w-\tilde{w}, (w_t - \operatorname{div}(D\nabla w)) - (\tilde{w}_t - \operatorname{div}(D\nabla\tilde{w}))\rangle_{X,X^*} \\ &= 2\left\{(w-\tilde{w}, h+u-\tilde{h}-\tilde{u})_H - (w-\tilde{w}, w-\tilde{w})_H\right\} \\ &\leq 2\left(\|w-\tilde{w}\|_X\|h+u-\tilde{h}-\tilde{u}\|_H - \|w-\tilde{w}\|_H^2\right) \\ &\leq \delta\|w-\tilde{w}\|_H^2 + \frac{1}{\delta}\|h+u-\tilde{h}-\tilde{u}\|_H^2 - 2\|w-\tilde{w}\|_H^2, \end{aligned}$$

where we used the continuous embeddings $X \subset H \subset X^*$ and Young's inequality with $\delta > 0$. Hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|w-\tilde{w}\|_H^2 &\leq (\delta-2)\|w-\tilde{w}\|_H^2 + \frac{1}{\delta}\|h+u-\tilde{h}-\tilde{u}\|_H^2 - 2\alpha\|\nabla w - \nabla\tilde{w}\|_H^2 \\ &\leq (\delta-2)\|w-\tilde{w}\|_H^2 + \frac{1}{\delta}\|h+u-\tilde{h}-\tilde{u}\|_H^2 + (\delta-2\alpha)\|\nabla w - \nabla\tilde{w}\|_H^2. \end{aligned}$$

Choosing $\delta = 2\min\{1,\alpha\}$ and using the global Lipschitz continuity (1.1.6), we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|w-\tilde{w}\|_H^2 &\leq C\|h+u-\tilde{h}-\tilde{u}\|_H^2 \\ &\leq C\left(\|F(u)-F(\tilde{u})\|_H^2 + \|u-\tilde{u}\|_H^2\right) \\ &\leq C(L^2+1)\,\|u-\tilde{u}\|_H^2, \end{aligned}$$

where $C = 1/(2\min\{1,\alpha\})$. Integrating over the time interval $(0,s)$ yields for $0 \leq s \leq T$

$$\begin{aligned} \|w(s)-\tilde{w}(s)\|_H^2 &= \int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\|w-\tilde{w}\|_H^2\,\mathrm{d}t \qquad (w(0) = \tilde{w}(0) = u_0) \\ &\leq C\int_0^s \|u(t)-\tilde{u}(t)\|_H^2\,\mathrm{d}t \leq CT\|u-\tilde{u}\|_V^2, \end{aligned}$$

where the constant $C \geq 0$ depends on the parameters α and L. Thus, we find

$$\|w-\tilde{w}\|_V^2 = \max_{0\leq s\leq T}\|w(s)-\tilde{w}(s)\|_H^2 \leq CT\|u-\tilde{u}\|_V^2$$

what from follows

$$\|\Phi(u) - \Phi(\tilde{u})\|_X \leq \sqrt{CT}\|u - \tilde{u}\|_V.$$

Choosing $T > 0$ such that $\sqrt{CT} < 1 \Leftrightarrow T < C^{-1}$, the map Φ is a contraction and Banach's fixed point theorem yields the existence of a unique solution $u = \Phi(u)$ of (1.1.9).

Step 2: Globality of solutions. We show that solutions do not blow up in finite time $T > 0$. Let u be a solution of (1.1.9) according to Step 1. We integrate the weak formulation over $(0,t)$ for $t \leq T$, test with $\varphi = u$, use the growth condition (1.1.8), apply Young's inequality with $\mu > 0$, and obtain

$$\begin{aligned}
&\frac{1}{2}(\|u(t)\|_H^2 - \|u(0)\|_H^2) = \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^t \left(\int_\Omega -D\nabla u : \nabla u + F(u)\cdot u \,\mathrm{d}x + \langle f, u\rangle\right) \mathrm{d}t \\
&\leq -\alpha\|\nabla u\|^2_{\mathrm{L}^2(0,t;H)} + \|C_1(1+|u|)^2\|_{\mathrm{L}^1(0,t;\mathrm{L}^1(\Omega))} + C_f\|u\|_{\mathrm{L}^2(0,t;X)} \\
&\leq C(C_1, |\Omega|, C_f)(1 + \|u\|^2_{\mathrm{L}^2(0,t;H)}) + \tfrac{C_f^2}{2\mu} + (\tfrac{1}{2}\mu - \alpha)\|\nabla u\|^2_{\mathrm{L}^2(0,t;H)}.
\end{aligned} \tag{1.1.11}$$

Choosing $\mu = 2\alpha$, there exists a constant $C \geq 0$ depending on $C_1, |\Omega|, C_f$, and the ratio C_f^2/α. Thus, we can apply Gronwall's inequality (integral version) to

$$\|u(t)\|_H^2 \leq \|u_0\|_H^2 + C\left(1 + \|u\|^2_{\mathrm{L}^2(0,t;H)}\right),$$

which yields $\|u(t)\|_H^2 + 1 \leq \|u_0\|_H^2(1 + CT\exp(TC))$ for all $0 \leq t \leq T$. Hence, there exists $c_1 = C(\|u_0\|_H, T, L, C_0, |\Omega|, C_f, C_f/\sqrt{\alpha}) \geq 0$ such that

$$\|u\|_V \leq c_1.$$

Moreover, the solution u is maximal, i.e. it exists for all positive T. In view of the boundedness of u up to time T, we can construct a weak solution u^1 on the interval $[T, 2T]$ with initial value $u_0^1 = u(T) \in H$ and repeating this argument gives the globality.

In the following two steps, we derive the boundedness of ∇u and u_t in (1.1.10).

Step 3: Choosing $\mu = \alpha$ in (1.1.11) yields

$$\tfrac{1}{2}\alpha\|\nabla u\|^2_{\mathrm{L}^2(0,t;H)} \leq C(c_1)(1 + \|u\|^2_{\mathrm{L}^2(0,t;H)} + \|u\|_V^2). \tag{1.1.12}$$

Since $t \in (0,T]$ was chosen arbitrarily, we obtain $\sqrt{\alpha}\|\nabla u\|_{\mathrm{L}^2(0,T;H)} \leq c_2 < \infty$ and c_2 depends on the same set of parameters as c_1.

Step 4: Analogously to Step 2, we obtain by applying Hölder's inequality and (1.1.8):

$$\begin{aligned}
\|u_t\|^2_{\mathrm{L}^2(0,T;X^*)} &= \int_0^T \left(\sup_{\|\varphi\|_X = 1} -(D\nabla u, \nabla\varphi)_H + \langle F(u) + f, \varphi\rangle_{X^*,X}\right)^2 \mathrm{d}t \\
&\leq \int_0^T \left(\sup_{\|\varphi\|_X = 1} \beta\|\nabla u\|_H\|\varphi\|_X + C_1^2(|\Omega| + \|u\|_H + \|f(t)\|_{X^*})\|\varphi\|_X\right)^2 \mathrm{d}t \\
&\leq 2\left(T\left(\beta\tfrac{1}{\sqrt{\alpha}}c_2 + C(c_1)\right)^2 + C_f^2\right),
\end{aligned}$$

where β is from (1.1.5). Hence Step 2–4 imply the existence of a constant C_a, depending on $\|u_0\|_H, T, L, C_\infty, |\Omega|, C_f, C_f/\sqrt{\alpha}$, such that (1.1.10) holds true. □

We complete this subsection with Proposition 1.1.3 that gives improved time-regularity for solutions u of (1.1.1.P), i.e.

$$W_{\mathrm{imp}}(0,T;X) := \mathrm{H}^1(0,T;X) \cap \mathrm{H}^2(0,T;X^*) \tag{1.1.13}$$

and, in particular $u_t \in C([0,T];H) \subsetneq \mathrm{L}^2(0,T;X^*)$. This is motivated by the fact that the folding and unfolding operators, defined in Subsection 1.2.2, are only well-defined for integrable functions. Since the diffusion term is degenerating in ($1.\mathrm{P}_\varepsilon^{\mathrm{RD}}$) as $\varepsilon \to 0$, we cannot apply the theory of maximal parabolic regularity to obtain $u_t(t) \in H$.

Proposition 1.1.3 (Improved time-regularity)**.** *Let the assumptions of Theorem 1.1.2 hold true. We assume the additional regularity for the initial value:*

$$\mathcal{A}(0)u_0 \in H. \tag{1.1.14}$$

Then, we have for all solutions u of (1.1.1.P) *that $u \in W_{\mathrm{imp}}(0,T;X)$ and it holds*

$$\|u\|_{C^1([0,T];H)} + \sqrt{\alpha}\|\nabla u\|_{\mathrm{H}^1(0,T;H)} + \|u_t\|_{\mathrm{H}^1(0,T;X^*)} \le C_a^* \left(1 + \tfrac{\beta}{\sqrt{\alpha}}\right). \tag{1.1.15}$$

Here, the constant $C_a^* \ge 0$ depends on C_a from (1.1.10) *as well as the initial condition $\|\mathcal{A}(0)u_0\|_H$ and the ratio $\|D_t\|_{C([0,T];\mathrm{L}^\infty(\Omega))}/\alpha$.*

Proof. We follow the idea of the proof to [Tem88, Thm. 3.2]. Let $u \in W(0,T;X)$ denote the unique solution to (1.1.1.P) according to Theorem 1.1.2. By setting $w = u_t$ and recalling $\mathcal{A}(t,x)u = \mathrm{div}(D(t,x)\nabla u)$, (formally) differentiating (1.1.1.P) w.r.t. time t gives

$$w_t = u_{tt} = \left(\mathrm{div}(D\nabla u) + F(u)\right)_t = \mathrm{div}(D_t\nabla u) + \mathrm{div}(D\nabla w) + F_t(u) + \mathrm{D}F(u)\cdot w.$$

This leads to a reaction-diffusion equation of the type (1.1.1.P), i.e.

$$w_t = \mathcal{A}w + \widetilde{F}(w) + f \quad \text{in } [0,T]\times\Omega \quad \text{with} \quad w(0) = u_t(0), \tag{1.1.16}$$
$$\text{where} \quad \widetilde{F}(t,x,A) := F_t(t,x,u(t,x)) + \mathrm{D}F(t,x,u(t,x))A \quad \text{and} \quad \widetilde{f}(t,x) := \mathrm{div}(D_t\nabla u).$$

Here, $\mathrm{D}F$ denotes the derivative of F with respect to A with $|\mathrm{D}F(t,x,A)| \le L$ for all $(t,x,A) \in [0,T]\times\Omega\times\mathbb{R}^m$. The function $\widetilde{F}$ immediately satisfies the Lipschitz continuity assumption (1.1.6), since it acts linear on A. Moreover, we have $D_t \in \mathrm{L}^\infty([0,T]\times\Omega)$ and $F_t(u) \in \mathrm{L}^2([0,T]\times\Omega)$ thanks to (1.1.5) and (1.1.6). Thus, it holds $\widetilde{f} \in \mathrm{L}^2(0,T;X^*)$ with

$$\|\widetilde{f}\|_{\mathrm{L}^2(0,T;X^*)} \le D_\infty\|\nabla u\|_{\mathrm{L}^2(0,T;H)} \le D_\infty\frac{C(c_1)}{\sqrt{\alpha}}$$

by setting $D_\infty := \|D_t\|_{C([0,T];\mathrm{L}^\infty(\Omega))}$ and $f = 0$ in (1.1.9) (as in (1.1.1.P)) as well as using (1.1.12). With this, the ratio $C_f/\sqrt{\alpha}$ becomes D_∞/α.

With (1.1.14), the initial value for w in (1.1.16) satisfies in $t = 0$

$$w(0) = u_t(0) = \mathrm{div}(D(0)\nabla u_0) + F(0,u_0) \in H.$$

Regardless that $\widetilde{F}$ is not differentiable with respect to (t,A), the necessary assumptions of [Tem88, Thm. 3.4] in Step 1 of the proof to Theorem 1.1.2 and the global Lipschitz continuity (1.1.6) are satisfied and we obtain the existence of a unique solution $w \in W(0,T;X)$ of (1.1.16). And hence, $u \in W_{\mathrm{imp}}(0,T;X) \subset C^1([0,T];H)$.

In order to rigorously justify $w = u_t$ we can argue as in [Wlo82, Satz 27.2]. □

1.1.3 Discussion of the assumptions

The additional assumption (1.1.14) seems to be restrictive on the initial value u_0 and on the diffusion tensor D, but actually D can be as general as in Theorem 1.1.2. We interpret (1.1.14) as a restriction on the choice of the initial value u_0, while D is possibly discontinuous in space. Indeed for $D \in \mathcal{M}(\Omega)$ and arbitrary $g \in H$, we can solve for $t = 0$ the static equation

$$\operatorname{div}(D(0)\nabla u_0) - u_0 = g \quad \text{in } \Omega \tag{1.1.17}$$

and we obtain by the Lax–Milgram lemma a unique solution $u_0 \in X$. In particular, we have $\mathcal{A}(0)u_0 = \operatorname{div}(D(0)\nabla u_0) \in H$.

We emphasize that the improved time-regularity, and therefore the more restrictive assumptions on F and D (differentiability) and u_0 (as in (1.1.17)), are only needed for technical reasons, i.e. the application of $\mathcal{T}_\varepsilon$.

Assuming further structural assumptions on F and u_0, one can prove even $\mathrm{L}^\infty(\Omega)$-estimates for the solutions, cf. e.g. [GlH97, Thm. 4.2], [BoH03, Lem. 1], [NeJ07, Lem. 3.1], [FMP12, Lem. 2.4], [GPS14, Lem. 4.1 & 4.2], or [Pie10, Lem. 1.1]. Such boundedness is meaningful, when u_i denotes a chemical concentration. In particular, it justifies the modification of the nonlinear reaction term outside a large ball and hence, the assumption of global Lipschitz continuity can be fulfilled easily.

Example 1.1.4 (A system with quadratic nonlinearity)**.** *We consider a system with two species X_u and X_v, with densities $u, v \geq 0$ interacting through one reaction of the type $X_u \rightleftharpoons 2X_v$. Normalizing the densities suitably, the mass-action law leads to the system*

$$u_t = \delta_u \Delta u + k\,(v^2 - u), \quad v_t = \delta_v \Delta v + 2k\,(u - v^2), \tag{1.1.18}$$

where $\delta_u, \delta_v > 0$ and the reaction coefficient k is given via $k(u,v) = \dfrac{k_0}{1+\alpha u+\beta v}$. The numerator $k_0 > 0$ denotes the empirical reaction rate and the denominator $1+\alpha u+\beta v$, for $0 < \alpha, \beta \ll 1$, leads to partial saturation of the reaction for large values of $u, v > 0$. The nonlinearity $F(u,v) = k(u,v)\begin{pmatrix} v^2 - u \\ 2(u - v^2) \end{pmatrix}$ is differentiable and globally Lipschitz continuous with constant $L = O(\max\{\frac{\alpha}{\beta^2}, \frac{1}{\beta}\})$. Hence, F satisfies the assumption (1.1.6). In many applications (cf. e.g. [Mie11, Mie13] for general reaction-diffusion systems based on the mass-action law) the reaction terms are given by polynomials and choosing suitable prefactors one obtains globally Lipschitz continuous $F \in \mathcal{F}(\Omega)$, e.g. the Shockley–Read–Hall term in semiconductor equations [MRS90, Eq. (3.1.9)] or in Michaelis–Menten kinetics for enzymatic catalysis [Mur02, pp. 175].

1.2 Two-scale convergence

In the introduction, the original model ($1.\mathrm{P}_\varepsilon^{\mathrm{RD}}$) is formulated on one scale, i.e. $x \in \Omega$, while the limit model ($2.\mathrm{P}_0^{\mathrm{RD}}$) is defined on the two-scale space $(x,y) \in \Omega \times \mathcal{Y}$. Here the microscopic variable y captures periodic oscillations in x/ε and x denotes the macroscopic variable. In order to rigorously derive homogenization results for ($1.\mathrm{P}_\varepsilon^{\mathrm{RD}}$) and ($6.\mathrm{P}_\varepsilon^{\mathrm{CH}}$), we introduce in this section the concept of two-scale convergence via periodic unfolding.

The concept of two-scale convergence is designed for periodic homogenization and roughly speaking it makes asymptotic expansion rigorous. The original definition, introduced in [Ngu89], reads: We say a sequence $(u_\varepsilon)_\varepsilon \subset \mathrm{L}^2(\Omega)$ is *converging in the two-scale sense* to a limit $U \in \mathrm{L}^2(\Omega \times \mathcal{Y})$, if

$$\int_\Omega u_\varepsilon(x)\Phi(x, \tfrac{x}{\varepsilon})\,\mathrm{d}x \to \int_{\Omega\times\mathcal{Y}} U(x,y)\Phi(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \Phi \in C_{\mathrm{c}}^\infty(\Omega\times\mathcal{Y}). \tag{1.2.1}$$

We point out that the choice of admissible test functions is not trivial, since $\Phi \in \mathrm{L}^2(\Omega\times\mathcal{Y})$ is not well-defined on the null-set $\{(x, x/\varepsilon)\} \subset \Omega\times\mathcal{Y}$, unless $y \mapsto \Phi(x,y)$ is continuous. For more details concerning the space of admissible test functions, we refer to [LNW02].

Whereas (1.2.1) describes a weak notion of convergence, it is necessary to use concepts based on a strong notion of convergence for the nonlinear problems ($1.\mathrm{P}_\varepsilon^{\mathrm{RD}}$) and ($6.\mathrm{P}_\varepsilon^{\mathrm{CH}}$). Demanding in (1.2.1) additionally the convergence of the norms, i.e. $\|u_\varepsilon\|_{\mathrm{L}^2(\Omega)} \to \|U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})}$, one may speak of strong convergence in the two-scale sense. However, this notion is not very handy. Therefore, inspired by the dilation operator in [ADH90], the *periodic unfolding operator* $\mathcal{T}_\varepsilon : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ was introduced in [CDG02]. With the aid of $\mathcal{T}_\varepsilon$, *weak and strong two-scale convergence* of $(u_\varepsilon)_\varepsilon$ is defined via classical weak and strong convergence of $(\mathcal{T}_\varepsilon\, u_\varepsilon)_\varepsilon$ in the two-scale space $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$.

Section 1.2 is structured as follows. We introduce the concept of the *periodicity cell* $\mathcal{Y}$ as well as the decomposition in macro- and microscopic scale in Subsection 1.2.1 and we define the unfolding and folding operators $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$ in Subsection 1.2.2. In Subsection 1.2.3, we give the definition of weak and strong two-scale convergence, and in Subsection 1.2.4, we focus on Sobolev functions.

1.2.1 Microstructure and periodicity cell

Following [CDG08, Sec. 2.1], let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $Y \subset \mathbb{R}^d$ denote the *unit cell*. Here and afterward, we set $Y = [0,1)^d$, but more general choices for Y are possible, see e.g. [MiT07, Sec. 2.1], so that $\mathbb{R}^d$ is the disjoint union of translated cells $\lambda + Y$, where $\lambda \in \mathbb{Z}^d$. Furthermore, we distinguish the unit cell Y from the periodicity cell $\mathcal{Y}$, which is obtained by identifying the opposite faces of $\overline{Y}$, i.e. the torus

$$\mathcal{Y} := \mathbb{R}^d/_{\mathbb{Z}^d}.$$

But, in notation, we will not distinguish between elements of the unit cell $y \in Y$ and the ones of the periodicity cell $y \in \mathcal{Y}$. Let $[x]_Y = (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$ denote the component-by-component application of the classical Gauss bracket. Using the mappings $[\,\cdot\,]_Y : \mathbb{R}^d \to \mathbb{Z}^d$ and $\{\,\cdot\,\}_Y : \mathbb{R}^d \to Y$ defined via the relation $x = [x]_Y + \{x\}_Y$, each point $x \in \mathbb{R}^d$ is uniquely decomposed into an element of the unit cell $\{x\}_Y \in Y$ and a lattice point $[x]_Y \in \mathbb{Z}^d$. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is called *Y-periodic*, if $f(x) = f(\{x\}_Y)$ for a.a. $x \in \mathbb{R}^d$. Then, we can identify every periodic function f with a function $\widetilde{f}$ on $\mathcal{Y}$. Whereas $\mathrm{L}^p(\mathcal{Y})$ and $\mathrm{L}^p(Y)$ can be identified, $W^{1,p}(\mathcal{Y}) = W^{1,p}_{\mathrm{per}}(Y)$ is a closed subspace of $W^{1,p}(Y)$.

For our multiscale problems ($1.\mathrm{P}_\varepsilon^{\mathrm{RD}}$) and ($6.\mathrm{P}_\varepsilon^{\mathrm{CH}}$), we introduce the small length-scale parameter $\varepsilon > 0$ and use the abbreviation

$$\mathcal{N}_\varepsilon(x) := \varepsilon \left[\tfrac{x}{\varepsilon}\right]_Y$$

for the nodes of the microscopic cells $\{\varepsilon(\lambda + Y) \mid \lambda \in \mathbb{Z}^d\}$, which describe the macroscopic scale. The microscopic scale is given by $y = \{\frac{x}{\varepsilon}\}_Y \in Y$ so that we obtain for all $x \in \mathbb{R}^d$ the decomposition $x = \mathcal{N}_\varepsilon(x) + \varepsilon y$ as shown in Figure 1.1.

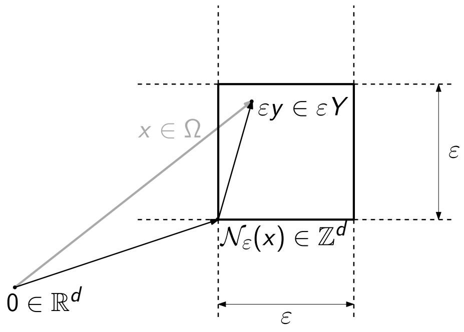

Figure 1.1: Decomposition into macroscopic $\mathcal{N}_\varepsilon(x)$ and microscopic y scale.

Since the domain Ω is bounded and not the whole $\mathbb{R}^d$, we have to treat the cells close to the boundary $\partial\Omega$ with care so that cells intersecting $\partial\Omega$ are sorted out for each $\varepsilon > 0$ fixed. We set

$$\Omega_\varepsilon^- := \operatorname{int}\Big(\bigcup\nolimits_{\lambda \in \Lambda_\varepsilon^-} \varepsilon(\lambda + Y)\Big) \quad \text{with} \quad \Lambda_\varepsilon^- := \{\lambda \in \mathbb{Z}^d \mid \varepsilon(\lambda + Y) \subset \overline{\Omega}\}. \tag{1.2.2}$$

Hence Ω_ε^- denotes (the interior of) the union of all microscopic cells $\varepsilon(\lambda + Y)$ contained in Ω. Similarly, we define the union of all microscopic cells that cover Ω via

$$\Omega_\varepsilon^+ := \operatorname{int}\Big(\bigcup\nolimits_{\lambda \in \Lambda_\varepsilon^+} \varepsilon(\lambda + Y)\Big) \quad \text{with} \quad \Lambda_\varepsilon^+ := \{\lambda \in \mathbb{Z}^d \mid \operatorname{int}(\varepsilon(\lambda + Y)) \cap \Omega \neq \emptyset\}. \tag{1.2.3}$$

With this, we clearly have $\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+$ as shown in Figure 1.2. For bounded domains

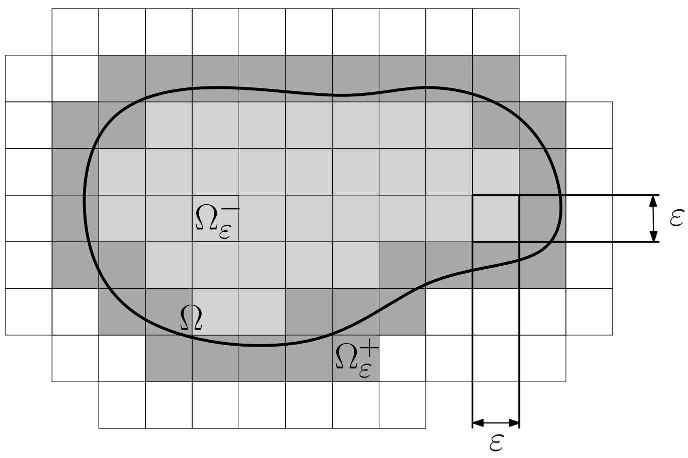

Figure 1.2: Covering of the domain Ω with microscopic cells.

Ω with Lipschitz boundary, we have by [Han11, Eq. (2.3)] that

$$\operatorname{vol}\Big(\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-\Big) \to 0. \tag{1.2.4}$$

1.2.2 Folding and periodic unfolding operators

Two-scale convergence is suited to describe convergences on different scales, namely the macroscopic scale, represented by $x \in \Omega$, and the microscopic scale for $y \in \mathcal{Y}$. Therefore the notion of a suitable embedding of the function space $\mathrm{L}^p(\Omega)$ into the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$ is desirable in order to find a "natural" definition of two-scale convergence. Here, we call such a mapping *periodic unfolding operator*. Vice versa, for any two-scale function U defined on $\Omega \times \mathcal{Y}$ we seek a one-scale dependent u_ε defined on Ω, and we call a corresponding mapping from the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$ into $\mathrm{L}^p(\Omega)$ *folding operator*.

Following [CDG02, CDG08, MiT07], the *periodic unfolding operator* $\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$ with $1 \leq p \leq \infty$ is defined via

$$(\mathcal{T}_\varepsilon\, u)(x,y) := u_{\mathrm{ex}}(\mathcal{N}_\varepsilon(x) + \varepsilon y), \tag{1.2.5}$$

where $u_{\mathrm{ex}} \in \mathrm{L}^p(\mathbb{R}^d)$ is obtained from u by extension with 0 outside of Ω. By definition, we have immediately the *product rule*

$$\mathcal{T}_\varepsilon(u_1 u_2) = (\mathcal{T}_\varepsilon\, u_1)(\mathcal{T}_\varepsilon\, u_2) \in \mathrm{L}^r(\mathbb{R}^d \times \mathcal{Y}) \quad \text{for all } u_i \in \mathrm{L}^{p_i}(\Omega) \text{ with } \tfrac{1}{p_1} + \tfrac{1}{p_2} \leq \tfrac{1}{r}. \tag{1.2.6}$$

Moreover, we obtain (see [Dam05, p. 121] or [HaK12, Eq. (5.2)]) the *integral identity*

$$\int_\Omega u \,\mathrm{d}x = \int_{\mathbb{R}^d \times \mathcal{Y}} \mathcal{T}_\varepsilon\, u \,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } u \in \mathrm{L}^1(\Omega). \tag{1.2.7}$$

With $[\Omega \times \mathcal{Y}]_\varepsilon := \{\, (x,y) \in \mathbb{R}^d \times \mathcal{Y} \mid \mathcal{N}_\varepsilon(x) + \varepsilon y \in \Omega \,\}$ we have $\operatorname{supp}(\mathcal{T}_\varepsilon\, u) \subseteq \overline{[\Omega \times \mathcal{Y}]_\varepsilon}$, i.e. in general the support of a two-scale function $\mathcal{T}_\varepsilon\, u$ is not contained in $\Omega \times \mathcal{Y}$. For a proper definition of the reverse operation taking care of the overhanging supports, we follow the construction of the folding operator in [MiT07], which involves the characteristic functions $\mathbb{1}_\Omega$ and $\mathbb{1}_\varepsilon := \mathcal{T}_\varepsilon\, \mathbb{1}_\Omega$ of Ω and $[\Omega \times \mathcal{Y}]_\varepsilon$, respectively. The *folding operator* $\mathcal{F}_\varepsilon : \mathrm{L}^q(\mathbb{R}^d \times \mathcal{Y}) \to \mathrm{L}^q(\Omega)$ with $1 \leq q < \infty$ is defined via

$$(\mathcal{F}_\varepsilon\, U)(x) := \left(\fint_{\mathcal{N}_\varepsilon(x) + \varepsilon Y} \mathbb{1}_\varepsilon(z, \{\tfrac{x}{\varepsilon}\}_Y) \cdot U(z, \{\tfrac{x}{\varepsilon}\}_Y) \,\mathrm{d}z \right)\Bigg|_\Omega . \tag{1.2.8}$$

Note that the identity $\mathrm{L}^q(\mathbb{R}^d \times \mathcal{Y}) = \mathrm{L}^q(\mathbb{R}^d; \mathrm{L}^q(\mathcal{Y}))$ holds for all $1 \leq q < \infty$. We will use several properties of $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$, see [MiT07, Prop. 2.1]:

Proposition 1.2.1. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $\varepsilon > 0$, we have the following properties:*

(a) $\|\, \mathcal{T}_\varepsilon\, u\|_{\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})} = \|u\|_{\mathrm{L}^p(\Omega)}$ *and* $\operatorname{supp}(\mathcal{T}_\varepsilon\, u) \subset \overline{[\Omega \times \mathcal{Y}]_\varepsilon}$ *for all* $u \in \mathrm{L}^p(\Omega)$.

(b) $\|\, \mathcal{F}_\varepsilon\, U\|_{\mathrm{L}^q(\Omega)} \leq \|U\|_{\mathrm{L}^q(\mathbb{R}^d \times \mathcal{Y})}$ *for all* $U \in \mathrm{L}^q(\mathbb{R}^d \times \mathcal{Y})$.

(c) $\mathcal{F}_\varepsilon \circ \mathcal{T}_\varepsilon = \mathrm{id}_{\mathrm{L}^q(\Omega)}$.

(d) $\mathcal{F}_\varepsilon$ *is the adjoint of* $\mathcal{T}_\varepsilon$, *i.e.* $\mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon{}'$.

Of course, (a) and (b) hold as well as with $p \in \{1, \infty\}$ and $p = 1$, respectively. The following result states in which sense the periodic unfolding operator $\mathcal{T}_\varepsilon$ is compatible with differentiation and composition of functions.

Theorem 1.2.2 (Properties of $\mathcal{T}_\varepsilon$)**.**

(a) Let $1 \le p < \infty$. For $u \in W^{1,p}(\Omega)$, we have $\mathcal{T}_\varepsilon u \in \mathrm{L}^p(\mathbb{R}^d; W^{1,p}(Y))$ and

$$\mathcal{T}_\varepsilon(\varepsilon\nabla u) = \nabla_y(\mathcal{T}_\varepsilon u).$$

(b) For $F \in \mathcal{F}(\Omega)$ and $u \in \mathrm{L}^2(\Omega)$ we have $\mathcal{T}_\varepsilon[F(u)] = \mathcal{T}_\varepsilon F(\mathcal{T}_\varepsilon u)$.

Proof. For part (a), we refer to [Dam05, Thm. 5.1]. Part (b) follows from (1.2.5), i.e.

$$\begin{aligned}\mathcal{T}_\varepsilon [F(u)](x,y) &= \begin{cases} F(\mathcal{N}_\varepsilon(x)+\varepsilon y, u(\mathcal{N}_\varepsilon(x)+\varepsilon y)), & \text{if } (x,y) \in [\Omega\times\mathcal{Y}]_\varepsilon \\ 0, & \text{if } (x,y) \in (\mathbb{R}^d\times\mathcal{Y})\backslash[\Omega\times\mathcal{Y}]_\varepsilon \end{cases} \\ &= F(\mathcal{N}_\varepsilon(x)+\varepsilon y, u(\mathcal{N}_\varepsilon(x)+\varepsilon y)_{\mathrm{ex}})_{\mathrm{ex}} = \mathcal{T}_\varepsilon F(\mathcal{T}_\varepsilon u)(x,y),\end{aligned}$$

which finishes the proof. □

We emphasize that the unfolded function $\mathcal{T}_\varepsilon u$ in Theorem 1.2.2(a) is not Y-periodic and we call this *periodicity defect* as in [Gri04], i.e.

$$\text{for } u \in W^{1,p}(\Omega): \quad \mathcal{T}_\varepsilon u \in \mathrm{L}^p(\mathbb{R}^d; W^{1,p}(Y)) \supsetneqq \mathrm{L}^p(\mathbb{R}^d; W^{1,p}(\mathcal{Y})). \tag{1.2.9}$$

1.2.3 Weak and strong two-scale convergence

We are now in the position to give the definition of weak and strong two-scale convergence following again [MiT07]. The notion of two-scale convergence was first introduced in [Ngu89] and coincides for bounded sequences with Definition 1.2.3(a), here below, and a more detailed comparison of the different definitions is given in [MiT07, Sec. 2.3]. Since the construction of the periodic unfolding operator was quite technical, the definition of weak and strong two-scale convergence can now be stated easily:

Definition 1.2.3 (Weak and strong two-scale convergence)**.** *Let $1 \le p < \infty$. For $(u_\varepsilon)_\varepsilon$ a sequence in $\mathrm{L}^p(\Omega)$*

(a) we say that u_ε weakly two-scale converges to U in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and we write "$u_\varepsilon \xrightharpoonup{2\mathrm{w}} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$", if $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U_{\mathrm{ex}}$ weakly in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$;

(b) we say that u_ε strongly two-scale converges to U in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and we write "$u_\varepsilon \xrightarrow{2\mathrm{s}} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$", if $\mathcal{T}_\varepsilon u_\varepsilon \to U_{\mathrm{ex}}$ strongly in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$.

Note that the weak and strong convergence is asked to occur in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$ and not in $\mathrm{L}^p(\Omega\times\mathcal{Y})$. Otherwise a slightly different notion of convergence is generated, see e.g. [MiT07, Ex. 2.3]. We denote the canonical embedding of one-scale functions into the space of two-scale functions by $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$ with $1 \le p \le \infty$, where

$$(Eu)(x,y) := u_{\mathrm{ex}}(x). \tag{1.2.10}$$

The following proposition collects various properties of two-scale convergence.

Proposition 1.2.4. *For $1 < p, q < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$ and for all $\varepsilon > 0$, we have the following properties:*

(a) $u_\varepsilon \xrightharpoonup{2\mathrm{w}} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y}) \implies \|u_\varepsilon\|_{\mathrm{L}^p(\Omega)}$ is bounded for all $\varepsilon > 0$.

(b) $u_\varepsilon \xrightharpoonup{2\mathrm{w}} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and $v_\varepsilon \xrightarrow{2\mathrm{s}} V$ in $\mathrm{L}^q(\Omega\times\mathcal{Y}) \implies \int_\Omega u_\varepsilon v_\varepsilon \,\mathrm{d}x \to \int_{\Omega\times\mathcal{Y}} UV \,\mathrm{d}x\,\mathrm{d}y$.

(c) *For all* $U \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ *there exists a sequence* $(u_\varepsilon)_\varepsilon$ *so that* $u_\varepsilon \xrightarrow{2\mathrm{s}} U$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y})$ *(for example* $u_\varepsilon = \mathcal{F}_\varepsilon U_{\mathrm{ex}}$*).*

(d) $u_\varepsilon \to u$ *in* $\mathrm{L}^p(\Omega) \implies u_\varepsilon \xrightarrow{2\mathrm{s}} Eu$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y})$.

(e) $u_\varepsilon \xrightharpoonup{2\mathrm{w}} U$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y}) \implies u_\varepsilon \rightharpoonup u$ *in* $\mathrm{L}^p(\Omega)$*, where* $u(x) = \int_{\mathcal{Y}} U(x,y)\,\mathrm{d}y$.

We refer to [MiT07, Prop. 2.4] for a proof of (a)–(d) and to [Dam05, Thm. 3.3] for (e).

Later on in Chapter 2, only the case $p=2$ is considered. For brevity, we introduce the function spaces

$$X := \mathrm{H}^1(\Omega),\ H := \mathrm{L}^2(\Omega),\ \mathbb{H} := \mathrm{L}^2(\Omega\times\mathcal{Y}),\ \text{and}\ \mathbb{H}_{\mathbb{R}^d} := \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y}) \tag{1.2.11}$$

and emphasize that the following results hold analogously for $1\le p<\infty$. The unfolding operator $\mathcal{T}_\varepsilon : H \to \mathbb{H}_{\mathbb{R}^d}$ is defined for the class of Lebesgue-integrable functions, where boundary values play no role, so that in particular $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y}) = \mathrm{L}^2(\mathbb{R}^d\times Y)$. In view of the *periodicity defect* (1.2.9), we carefully distinguish the spaces $\mathrm{H}^1(Y)$ and $\mathrm{H}^1(\mathcal{Y}) = \mathrm{H}^1_{\mathrm{per}}(Y)$, where the latter one is a closed subspace of $\mathrm{H}^1(Y)$. Moreover, we define

$$\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}) := \left\{ u \in \mathrm{H}^1(\mathcal{Y}) \mid \int_{\mathcal{Y}} u(y)\,\mathrm{d}y = 0 \right\}. \tag{1.2.12}$$

The gradient (with respect to $x\in\Omega$) of one-scale functions u is always denoted with ∇u, while for two-scale functions U the gradient $\nabla_x U$ with respect to $x\in\Omega$ and $\nabla_y U$ with respect to $y\in\mathcal{Y}$ are distinguished in notation.

The following theorem states the fundamental results for two-scale convergence, in particular part (b) is crucial for the strong two-scale convergence of the slowly diffusing species v_ε in (1.P$_\varepsilon^{\mathrm{RD}}$).

Theorem 1.2.5 (Compactness). *Let* $(u_\varepsilon)_\varepsilon$ *be a sequence of functions.*

(a) *If* $(u_\varepsilon)_\varepsilon \subset H$ *and* $\|u_\varepsilon\|_H \le C$*, then there exists* $U\in\mathbb{H}$ *and a subsequence* ε' *of* ε *such that it holds* $u_{\varepsilon'} \xrightharpoonup{2\mathrm{w}} U$ *in* $\mathbb{H}$.

(b) *If* $(u_\varepsilon)_\varepsilon \subset X$ *and* $\|u_\varepsilon\|_H + \varepsilon\|\nabla u_\varepsilon\|_H \le C$*, then there exists* $U \in \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ *and a subsequence* ε' *of* ε *such that* $u_{\varepsilon'} \xrightharpoonup{2\mathrm{w}} U$ *and* $\varepsilon'\nabla u_{\varepsilon'} \xrightharpoonup{2\mathrm{w}} \nabla_y U$ *in* $\mathbb{H}$.

(c) *If* $(u_\varepsilon)_\varepsilon \subset X$ *and* $\|u_\varepsilon\|_X \le C$*, then there exists* $u\in X$*, a two-scale function* $U \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$*, and a subsequence* ε' *of* ε *such that* $u_{\varepsilon'} \rightharpoonup u$ *in* X *and* $\nabla u_{\varepsilon'} \xrightharpoonup{2\mathrm{w}} \nabla u + \nabla_y U$ *in* $\mathbb{H}$.

Proof. For the proof of (a), we refer to [Ngu89], alternatively one can apply Prop. 1.2.1(a) and Banach's selection principle. Items (b) and (c) are shown in e.g. [All92, Prop. 1.14] or [Dam05, Thm. 5.2, Thm. 5.4]. For another scaling such as ε^γ with $0\le\gamma<\infty$, we refer to [PeB08, Thm. 3.4]. □

It is a well-known fact (cf. [Ngu89, Thm. 3], [All92, Prop. 1.14], [Vis04, Thm. 6.1]) that the two-scale limit U of a sequence $(u_\varepsilon)_\varepsilon$ is Y-periodic, although the unfolded sequence $(\mathcal{T}_\varepsilon u_\varepsilon)_\varepsilon$ is in general not Y-periodic, see in particular [CDG02, Prop. 3] and [Dam05, Thm. 5.2] for a proof in the periodic unfolding formulation. We call this observation $\mathcal{T}_\varepsilon$-*property of recovered periodicity* as in [MRT14], i.e.

$$\begin{aligned} \text{for all } (u_\varepsilon)_\varepsilon \subset X: \quad & \mathcal{T}_\varepsilon\, u_\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1(Y)) \not\subseteq \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y})), \text{ but} \\ & \text{w-}\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon\, u_\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y})), \text{ if the limit exists.} \end{aligned} \tag{1.2.13}$$

The two-scale spaces $\mathrm{H}^1(\mathcal{Y};\mathrm{L}^2(\Omega))$ and $\mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ can both be identified with the Hilbert space $\{U \in \mathrm{L}^2(\Omega\times\mathcal{Y}) \,|\, \nabla_y U \in \mathrm{L}^2(\Omega\times\mathcal{Y})\}$, supplemented with the scalar product $\int_{\Omega\times\mathcal{Y}} U\cdot V + \nabla_y U : \nabla_y V \,\mathrm{d}x\,\mathrm{d}y$, thanks to the isomorphism I mapping $\mathrm{L}^2(\Omega\times\mathcal{Y}) = \mathrm{L}^2(\Omega;\mathrm{L}^2(\mathcal{Y}))$ to $\mathrm{L}^2(\mathcal{Y}\times\Omega) = \mathrm{L}^2(\mathcal{Y};\mathrm{L}^2(\Omega))$ via $(IU)(y,x) := U(x,y)$. Based on the convergence results (b)–(c) of the previous theorem, we define (additionally to (1.2.11)) the spaces

$$\begin{aligned} \mathbb{X} := \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y})),\ \mathbb{X}_0 := \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})), \text{ and} \\ X_\varepsilon := X \text{ equipped with the norm } \|v\|_{X_\varepsilon} := \|v\|_H + \varepsilon\|\nabla v\|_H. \end{aligned} \tag{1.2.14}$$

The space X_ε reflects the degeneracy of the slow diffusing species v_ε in ($1.\mathrm{P}^{\mathrm{RD}}_\varepsilon$) and the two-scale spaces $\mathbb{X}$ and $\mathbb{X}_0$ arise in the formulation of the limiting system ($2.\mathrm{P}^{\mathrm{RD}}_0$). Here, the index 0 indicates the 0-order of degeneracy in the sense of ε^0, whereas $\mathbb{X} = \mathbb{X}_1$ represents the 1-order of degeneracy with ε^1. In this context, we also refer to Definition 1.2.7 for the gradient folding operator $\mathcal{G}^\gamma_\varepsilon$ with $\gamma \in \{0,1\}$.

We finish this subsection by stating two results, needed in the proof of Main Theorem I, concerning the multiplication and composition of sequences in $\mathbb{H}$.

Lemma 1.2.6 (Multiplication and composition of sequences in $\mathbb{H}$)**.** *Let $\varepsilon > 0$.*

(a) Let $(U_\varepsilon)_\varepsilon \subset \mathbb{H}$ with $U_\varepsilon \to U$ in $\mathbb{H}$ and $(M_\varepsilon)_\varepsilon \subset \mathrm{L}^\infty(\Omega\times\mathcal{Y})$ such that $\|M_\varepsilon\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})} \le C$ for some constant $C > 0$ and $M_\varepsilon(x,y) \to M(x,y)$ for almost every $(x,y) \in \Omega\times\mathcal{Y}$. Then, it holds $M_\varepsilon U_\varepsilon \to MU$ in $\mathbb{H}$.

(b) Let $F_\varepsilon \in \mathcal{F}(\Omega)$, $\mathbb{F} \in \mathcal{F}(\Omega\times\mathcal{Y})$ and $t \in (0,T)$ fixed. If for all vectors $A \in \mathbb{R}^m$ it is $F_\varepsilon(t,A) \xrightarrow{2\mathrm{s}} \mathbb{F}(t,A)$ in $\mathbb{H}$, then, for all $U \in \mathbb{H}$, we have $\mathcal{T}_\varepsilon\, F_\varepsilon(t,U) \to \mathbb{F}(t,U)$ in $\mathbb{H}$.

Proof. *Ad (a):* Extracting from $(U_\varepsilon)_\varepsilon$ a pointwisely convergent subsequence, we find that $M_\varepsilon U_\varepsilon \to MU$ pointwise a.e. in $\Omega\times\mathcal{Y}$ for this subsequence. Moreover, since $|M_\varepsilon U_\varepsilon| \le C|U_\varepsilon|$ a.e. in $\Omega\times\mathcal{Y}$ by assumption, the sequence $(CU_\varepsilon)_\varepsilon$ serves as an L^2–convergent majorant. Thus, Pratt's theorem, see [Els02, Thm. 5.1 p. 260], a variant of the dominated convergence theorem, yields the strong L^2–convergence of the subsequence. Arguing by contradiction for a different subsequence and by the uniqueness of the limit, we conclude the convergence of the *whole* sequence.

Ad (b): For shorter notation we omit indicating the t-dependence of the functions. We approximate $U_{\mathrm{ex}} \in \mathbb{H}_{\mathbb{R}^d}$ with a sequence of integrable step functions $\overline{U}_n = \sum_{i=1}^n \mathbb{1}_{U_i}\cdot A_i$, where $A_i \in \mathbb{R}^m$ and $\overline{\Omega}\times\mathcal{Y} \subset \bigcup_{i=1}^n U_i$, which exists since $\overline{\Omega}\times\mathcal{Y}$ is compact. Hence $\overline{U}_n \to U_{\mathrm{ex}}$ in $\mathbb{H}_{\mathbb{R}^d}$ and it follows by assumption that $\mathcal{T}_\varepsilon\, F_\varepsilon(\overline{U}_n) = \sum_{i=1}^n \mathbb{1}_{U_i}\cdot\mathcal{T}_\varepsilon\, F_\varepsilon(A_i) \xrightarrow{\varepsilon\to 0} \sum_{i=1}^n \mathbb{1}_{U_i}\cdot\mathbb{F}_{\mathrm{ex}}(A_i) = \mathbb{F}_{\mathrm{ex}}(\overline{U}_n)$ in $\mathbb{H}_{\mathbb{R}^d}$. Exploiting the Lipschitz continuity (1.1.6) and introducing suitable nils, i.e.

$$\begin{aligned} &\mathcal{T}_\varepsilon\, F_\varepsilon(U_{\mathrm{ex}}) - \mathbb{F}_{\mathrm{ex}}(U_{\mathrm{ex}}) \\ &= [\mathcal{T}_\varepsilon\, F_\varepsilon(U_{\mathrm{ex}}) - \mathcal{T}_\varepsilon\, F_\varepsilon(\overline{U}_n)] + [\mathcal{T}_\varepsilon\, F_\varepsilon(\overline{U}_n) - \mathbb{F}_{\mathrm{ex}}(\overline{U}_n)] + [\mathbb{F}_{\mathrm{ex}}(\overline{U}_n) - \mathbb{F}_{\mathrm{ex}}(U_{\mathrm{ex}})], \end{aligned}$$

we obtain $\| \mathcal{T}_\varepsilon F_\varepsilon(U_{\text{ex}}) - \mathbb{F}_{\text{ex}}(U_{\text{ex}})\|_{\mathbb{H}_{\mathbb{R}^d}} \xrightarrow{\varepsilon\to 0} 0$. □

1.2.4 Gradient folding and two-scale convergence of Sobolev functions

Even for smooth functions $U : \Omega \times \mathcal{Y} \to \mathbb{R}$ the folded function $\mathcal{F}_\varepsilon U$ is only piecewise constant in x, hence $\nabla(\mathcal{F}_\varepsilon U)$ cannot be determined in the classical sense. Therefore, we now define a so-called *gradient folding operator* $\mathcal{G}_\varepsilon^\gamma$, which assigns to each differentiable two-scale function $U \in \mathrm{H}^1(\Omega \times \mathcal{Y})$ a one-scale function $u_\varepsilon \in \mathrm{H}^1(\Omega)$. The definition of the above mentioned gradient folding operator $\mathcal{G}_\varepsilon^\gamma$ follows [Han11], where $0 \le \gamma < \infty$. There, the operator $\mathcal{G}_\varepsilon^\gamma$ is constructed via $\mathcal{T}_\varepsilon$ and various projections, but then it is shown that $\mathcal{G}_\varepsilon^\gamma$ is uniquely characterized by solving a linear elliptic PDE, see [Han11, Prop. 2.11] which is based on [Vis04, Thm. 6.1] and [MiT07, Prop. 2.10] for $\gamma = 1$.

Definition 1.2.7 (Gradient folding operator $\mathcal{G}_\varepsilon^\gamma$)**.** *Let $\varepsilon > 0$.*

$\gamma = 0:$ *The gradient folding operator $\mathcal{G}_\varepsilon^0 : \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\text{av}}(\mathcal{Y})) \to \mathrm{H}^1(\Omega)$ maps a pair of functions $(u, U) \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\text{av}}(\mathcal{Y}))$ to $u_\varepsilon := \mathcal{G}_\varepsilon^1(u, U)$, where $u_\varepsilon \in \mathrm{H}^1(\Omega)$ is the unique solution of the elliptic problem*

$$\int_\Omega (u_\varepsilon - u)\cdot\varphi + (\nabla u_\varepsilon - \mathcal{F}_\varepsilon[E\nabla u + \nabla_y U_{\text{ex}}]) : \nabla\varphi \,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathrm{H}^1(\Omega). \tag{1.2.15}$$

$\gamma = 1:$ *The gradient folding operator $\mathcal{G}_\varepsilon^1 : \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y})) \to \mathrm{H}^1(\Omega)$ maps a two-scale function $U \in \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y}))$ to $u_\varepsilon := \mathcal{G}_\varepsilon^1 U$, where $u_\varepsilon \in \mathrm{H}^1(\Omega)$ is the unique solution of the elliptic problem*

$$\int_\Omega (u_\varepsilon - \mathcal{F}_\varepsilon U_{\text{ex}})\cdot\varphi + (\varepsilon\nabla u_\varepsilon - \mathcal{F}_\varepsilon[\nabla_y U_{\text{ex}}]) : \varepsilon\nabla\varphi \,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathrm{H}^1(\Omega). \tag{1.2.16}$$

Let us explain why Definition 1.2.7 is well-defined, exemplary for $\gamma = 1$. While $\mathcal{F}_\varepsilon : \mathbb{H}_{\mathbb{R}^d} \to H$, we have $\mathcal{G}_\varepsilon^1 : \mathbb{X} \to X$. Thus the domains of the two operators differ not only with respect to the regularity of the admissible functions, but also with respect to the underlying domains for the space variable x, i.e. $x \in \mathbb{R}^d$ versus $x \in \Omega$. However, since both operators require L^2–regularity in x only, extending $U \in \mathbb{X}$ by 0 outside of Ω yields $U_{\text{ex}} \in \mathbb{X}_{\mathbb{R}^d} := \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1(\mathcal{Y}))$. Thus, $\mathcal{F}_\varepsilon U_{\text{ex}}$ indeed is well-defined in (1.2.16). In particular, $\mathcal{F}_\varepsilon U_{\text{ex}}$ and $\mathcal{F}_\varepsilon[\nabla_y U]_{\text{ex}} \in H$ can be understood as linear operators acting on U and moved, as inhomogeneities for the determination of u_ε, to the right-hand side of (1.2.16). Thus for $\varepsilon > 0$ fixed, the Lax–Milgram lemma yields the existence of a unique solution $u_\varepsilon \in X$, so that the gradient folding operator $\mathcal{G}_\varepsilon^1$ is indeed well-defined. With the same arguments, $\mathcal{G}_\varepsilon^0 : X \times \mathbb{X}_0 \to X$ is well-defined, too.

Since (1.2.16) implies $\| \mathcal{G}_\varepsilon^1 U\|_{X_\varepsilon} \le C$, Theorem 1.2.5(b) supplies the existence of a weakly two-scale convergent subsequence. However, for given $U \in \mathbb{X}$ the gradient folding operator guarantees even *strong* two-scale convergence. Since $(\mathcal{G}_\varepsilon^1 U)_\varepsilon \subset X$ *recovers* any function $U \in \mathbb{X}$ via strong two-scale convergence, $\mathcal{G}_\varepsilon^\gamma$ is also called *recovery operator* in [Han11, pp. 10-12].

Proposition 1.2.8 (Recovery property [Han11, Prop. 2.11])**.** *Let $\varepsilon \to 0$.*

$\gamma = 0:$ *For all pairs of functions $(u, U) \in X \times \mathbb{X}_0$, we have $\mathcal{G}_\varepsilon^0(u, U) \rightharpoonup u$ in X and $\nabla[\mathcal{G}_\varepsilon^0(u, U)] \xrightarrow{2\text{s}} E\nabla u + \nabla_y U$ in $\mathbb{H}$.*

$\gamma = 1:$ *For all* $U \in \mathbb{X}$, *we have* $\mathcal{G}_\varepsilon^1 U \xrightarrow{2s} U$ *in* $\mathbb{H}$ *and* $\varepsilon\nabla[\mathcal{G}_\varepsilon^1 U] \xrightarrow{2s} \nabla_y U$ *in* $\mathbb{H}$.

Later on, in the proof of Main Theorem I, it will be essential to interchange differentiation and folding of two-scale functions $U \in \mathbb{X}$. However, convenient commutation relations, such as $\mathcal{F}_\varepsilon(\nabla_y U_{\mathrm{ex}}) = \varepsilon\nabla(\mathcal{F}_\varepsilon U_{\mathrm{ex}})$ or $\mathcal{G}_\varepsilon^1(\nabla_y U) = \varepsilon\nabla(\mathcal{G}_\varepsilon^1 U)$, cannot be expected, since $\mathcal{F}_\varepsilon U \notin X$ and $\nabla_y U \notin \mathbb{X}$. Instead, we establish a kind of commutation between $\mathcal{F}_\varepsilon(\nabla_y U_{\mathrm{ex}})$ and $\varepsilon\nabla(\mathcal{G}_\varepsilon^1 U)$. More precisely, the following result shows that $\mathcal{F}_\varepsilon U_{\mathrm{ex}}$ and $\mathcal{G}_\varepsilon^1 U$ are comparable in the sense that their difference vanishes.

Proposition 1.2.9 (Comparison of $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^\gamma$)**.** *Let* $\varepsilon \to 0$.
$\gamma = 0:$ *For all* $(u, U) \in X \times \mathbb{X}_0$, *we have*

$$\|u - \mathcal{G}_\varepsilon^0(u,U)\|_H + \|\mathcal{F}_\varepsilon[E\nabla u + \nabla_y U_{\mathrm{ex}}] - \nabla[\mathcal{G}_\varepsilon^0(u,U)]\|_H \to 0.$$

$\gamma = 1:$ *For all* $U \in \mathbb{X}$, *we have*

$$\|\mathcal{F}_\varepsilon U_{\mathrm{ex}} - \mathcal{G}_\varepsilon^1 U\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U)_{\mathrm{ex}} - \varepsilon\nabla(\mathcal{G}_\varepsilon^1 U)\|_H \to 0.$$

Proof. For $\gamma = 1$, we have by the triangle inequality

$$\begin{aligned}
&\|\mathcal{F}_\varepsilon U_{\mathrm{ex}} - \mathcal{G}_\varepsilon^1 U\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U)_{\mathrm{ex}} - \varepsilon\nabla(\mathcal{G}_\varepsilon^1 U)\|_H \\
&\le \|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon U_{\mathrm{ex}} - U_{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} + \|U_{\mathrm{ex}} - \mathcal{T}_\varepsilon\mathcal{G}_\varepsilon^1 U\|_{\mathbb{H}_{\mathbb{R}^d}} + \|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon(\nabla_y U)_{\mathrm{ex}} - \nabla_y U_{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} \\
&\quad + \|\nabla_y U_{\mathrm{ex}} - \mathcal{T}_\varepsilon[\varepsilon\nabla(\mathcal{G}_\varepsilon^1 U)]\|_{\mathbb{H}_{\mathbb{R}^d}} \xrightarrow{\varepsilon\to 0} 0,
\end{aligned}$$

by Proposition 1.2.4(d) for the terms involving $\mathcal{F}_\varepsilon$ and by Proposition 1.2.8 for the terms involving $\mathcal{G}_\varepsilon^1$. Analogously, we only split the gradient term in the case $\gamma = 0$

$$\begin{aligned}
&\|u - \mathcal{G}_\varepsilon^0(u,U)\|_H + \|\mathcal{F}_\varepsilon[E\nabla u + \nabla_y U_{\mathrm{ex}}] - \nabla[\mathcal{G}_\varepsilon^0(u,U)]\|_H \\
&\le \|u - \mathcal{G}_\varepsilon^0(u,U)\|_H + \|[E\nabla u + \nabla_y U_{\mathrm{ex}}] - \mathcal{T}_\varepsilon(\nabla[\mathcal{G}_\varepsilon^0(u,U)])\|_{\mathbb{H}_{\mathbb{R}^d}} \\
&\quad + \|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon[E\nabla u + \nabla_y U_{\mathrm{ex}}] - [E\nabla u + \nabla_y U_{\mathrm{ex}}]\|_{\mathbb{H}_{\mathbb{R}^d}} \xrightarrow{\varepsilon\to 0} 0,
\end{aligned}$$

where we have additionally used that $\mathcal{G}_\varepsilon^0(u,U) \rightharpoonup u$ in X implies $\mathcal{G}_\varepsilon^0(u,U) \to u$ in H thanks to the compact embedding $\mathrm{H}^1(\Omega) \subset \mathrm{L}^2(\Omega)$. $\square$

2 Two-scale homogenization of reaction-diffusion systems involving different diffusion length scales

We consider a system of two nonlinearly coupled reaction-diffusion systems, where the nonlinearity arises with the reaction term $(F_1^\varepsilon, F_2^\varepsilon)$, whereas the diffusion tensor has block structure, namely for $\varepsilon > 0$

$$\begin{pmatrix} u_t^\varepsilon \\ v_t^\varepsilon \end{pmatrix} = \begin{pmatrix} \operatorname{div}(D_1^\varepsilon \nabla u^\varepsilon) \\ \operatorname{div}(\varepsilon^2 D_2^\varepsilon \nabla v^\varepsilon) \end{pmatrix} + \begin{pmatrix} F_1^\varepsilon(u^\varepsilon, v^\varepsilon) \\ F_2^\varepsilon(u^\varepsilon, v^\varepsilon) \end{pmatrix} \quad \text{in } [0,T] \times \Omega. \tag{2.0.1.P$_\varepsilon$}$$

Throughout Chapter 2, let $\Omega \subset \mathbb{R}^d$ denote a bounded domain with Lipschitz boundary. We supplement (2.0.1.P$_\varepsilon$) with homogenous Neumann boundary conditions on $\partial\Omega$ and prescribed initial values $u^\varepsilon(0) = u_0^\varepsilon$ respective $v^\varepsilon(0) = v_0^\varepsilon$. Here, $D_i^\varepsilon \in \mathbb{R}^{(m_i \times d) \times (m_i \times d)}$ are diffusion tensors and F_i^ε are reaction terms acting on the vectors of concentrations $u^\varepsilon \in \mathbb{R}^{m_1}$ and $v^\varepsilon \in \mathbb{R}^{m_2}$ referring to m_1 and m_2 different species, respectively. We recall the abbreviation u_t^ε for the partial time derivative $\partial_t u^\varepsilon$ and, hence, we write ε as upper (and not lower) index throughout this chapter. We emphasize that the systems under consideration do not generally admit a gradient structure.

The parameter ε denotes the ratio between the characteristic macroscopic length scale such as the diameter of the domain Ω and the characteristic microscopic length scale of the underlying microstructure. This microstructure is encoded in the given data D_i^ε and F_i^ε which are (not exactly) periodic with respect to the micro-cell εY. The scaling ε^2 of D_2^ε takes into account that the species related to the concentration vector v^ε diffuse much slower than those related to u^ε. Therefore, we call v^ε the *slowly diffusing* variable and u^ε the *classically diffusing* one. We also call (2.0.1.P$_\varepsilon$)$_1$ the non-degenerating part while (2.0.1.P$_\varepsilon$)$_2$ is called the degenerating part.

We study the solutions $(u^\varepsilon, v^\varepsilon) : [0,T] \times \Omega \to \mathbb{R}^{m_1+m_2}$ of (2.0.1.P$_\varepsilon$) and their limit as ε tends to 0. While focusing on the derivation of effective equations, we avoid questions concerning the global existence and positivity of solutions. Throughout Chapter 2, we assume that the diffusion tensors D_i^ε are *uniformly elliptic and bounded* and that the reaction terms F_i^ε are *globally Lipschitz continuous and differentiable.* Moreover, the given data may depend on time, however for simplicity it can be assumed that their time-dependence is continuously differentiable.

We prove that $(u^\varepsilon, v^\varepsilon)$ converges for $\varepsilon \to 0$ to a limit (u, V) that decomposes into a one-scale function $u(t,x)$ and a two-scale function $V(t,x,y)$, which solve the effective system

$$\begin{pmatrix} u_t \\ V_t \end{pmatrix} = \begin{pmatrix} \operatorname{div}(D_{\text{eff}} \nabla u) \\ \operatorname{div}_y(\mathbb{D}_2 \nabla_y V) \end{pmatrix} + \begin{pmatrix} F_{\text{eff}}(u, V) \\ \mathbb{F}_2(u, V) \end{pmatrix} \quad \text{in } [0, T] \times \Omega \times \mathcal{Y}. \tag{2.0.2.P$_0$}$$

Here, the effective diffusion tensor D_{eff} and the effective u-reaction F_{eff} only depend on the macroscopic variable $x \in \Omega$. Whereas, the tensor D_{eff} is obtained via solving the well-known unit cell problem on $\mathcal{Y}$, see (2.1.16)–(2.1.17), F_{eff} is the usual average on $\mathcal{Y}$, namely the function-to-function map $F_{\text{eff}} : \Omega \times \mathbb{R}^{m_1} \times \mathrm{L}^2(\mathcal{Y}; \mathbb{R}^{m_2}) \to \mathbb{R}^{m_1}$ is defined as

$$F_{\text{eff}}(x, u, Z) := \fint_{\mathcal{Y}} \mathbb{F}_1(x, y, u, Z(y)) \, \mathrm{d}y. \tag{2.0.3}$$

Therefore, considering the u-equations $(2.0.2.\mathrm{P}_0)_1$ on their own, the solution u solves again a parabolic problem on $[0, T] \times \Omega$. In contrast, the diffusion tensor $\mathbb{D}_2$ and the V-reaction $\mathbb{F}_2$ depend on the two-scale variables $(x, y) \in \Omega \times \mathcal{Y}$, see assumption (2.1.12.Exist$_0$). One may interpret the V-equations in (2.0.2.P$_0$) as a parabolic problem on $[0, T] \times \mathcal{Y}$ and $x \in \Omega$ as a parameter.

The proof of the strong two-scale convergence

$$\| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\text{ex}}(t) \|_{\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})} \xrightarrow{\varepsilon \to 0} 0 \quad \text{uniformly in } [0, T], \tag{2.0.4}$$

is the *crucial* task. (As with ε, the index ex is written as upper index throughout this chapter.) A priori we only obtain weak two-scale convergence of v^ε, however we need strong convergence in order to pass to the limit $\varepsilon \to 0$ with the nonlinear reaction terms. For a discussion on the problem of slow diffusion and the accompanying loss of compactness and related results in the literature, we refer to the beginning of Section 2.1 and to Subsection 2.1.1, respectively. In Subsection 2.1.4, our approach, the occurring difficulties (such as the periodicity defect) and how we overcome those difficulties, are elaborated in more detail.

Chapter 2 contains three main theorems whereby all of them rely on the estimate

$$\begin{aligned} &\max_{0 \le t \le T} \left\{ \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\text{ex}}(t) \|^2_{\mathbb{H}_{\mathbb{R}^d}} + \| u^\varepsilon(t) - u(t) \|^2_H \right\} \\ &\le C \left\{ \| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\text{ex}} \|^2_{\mathbb{H}_{\mathbb{R}^d}} + \| u_0^\varepsilon - u_0 \|^2_H + \int_0^T \Delta^{v^\varepsilon}(t) + \Delta^{u^\varepsilon}(t) \, \mathrm{d}t \right\}. \end{aligned} \tag{2.0.5.Est}$$

Here, Δ^{w^ε} with $w^\varepsilon \in \{u^\varepsilon, v^\varepsilon\}$ comprises different error terms such as the *folding mismatch* $\Delta_1^{w^\varepsilon}$, the *periodicity defect error* $\Delta_2^{w^\varepsilon}$ and various *approximation errors* $\Delta_{3,4,5}^{w^\varepsilon}$. The derivation of (2.0.5.Est) in Theorem 2.1.6 relies on the application of Gronwall's lemma and on the periodic unfolding method. To apply the periodic unfolding operator $\mathcal{T}_\varepsilon$, the partial time derivatives $v_t^\varepsilon(t)$ and $V_t(t)$ need to be integrable functions.

Main Theorem I (Theorem 2.1.1) is devoted to the derivation of (2.0.2.P$_0$) as a set of effective equations for the original macroscopic system (2.0.1.P$_\varepsilon$) when the characteristic length scale ε tends to 0. Therefore, we assume in (2.1.14.Conv) that the given data converge suitably in the two-scale sense, namely $\mathcal{T}_\varepsilon D_i^\varepsilon(x, y) \to \mathbb{D}_i(x, y)$ pointwise as well as $\mathcal{T}_\varepsilon F_i^\varepsilon \to \mathbb{F}_i$ and $\mathcal{T}_\varepsilon v_0^\varepsilon \to V_0$ respective $u_0^\varepsilon \to u_0$ strongly in $\mathrm{L}^2(\Omega \times \mathcal{Y})$ respective $\mathrm{L}^2(\Omega)$. For technical reasons, we presume additional regularity of the initial values which guarantees improved time-regularity of the solutions for all times. Based on this, we derive estimate (2.0.5.Est) and control the error terms Δ^{v^ε} and Δ^{u^ε} as $\varepsilon \to 0$ so that the strong two-scale convergence (2.0.4) can be obtained. This result (in combination with Chapter 1) is based on the article [MRT14].

In Main Theorem II (Theorem 2.1.1), we relax the assumptions on the initial values and prove the homogenization of (2.0.1.P_ε) without improved time-regularity of the solutions. Therefore, it can be shown that general L^2-initial values can be approximated with more regular initial values such that the associated solutions satisfy improved time-regularity. The compatibility of homogenization and regularization follows straight forward.

In Main Theorem IIIa–b (Theorem 2.3.1 and 2.3.2), we quantify the convergence of the solutions $(u^\varepsilon, v^\varepsilon)$ to (u, V). Therefore, we assume higher regularity for the given data and the effective solution (u, V) with respect to $x \in \Omega$ and prove quantitative estimates for the error terms Δ^{v^ε} and Δ^{u^ε}. With this at hand as well as suitable convergence of the initial values, the explicit rate ε^η for the convergence of $(u^\varepsilon, v^\varepsilon)$ to (u, V) is obtained in (2.0.5.Est). Depending on the choice of the initial values, we have $\eta = \frac{1}{4}$ or $\eta = \frac{1}{6}$. In the case of slow diffusion only, interior estimates are available with $\eta = \frac{1}{2}$ or $\eta = \frac{1}{3}$, see Subsection 2.3.6. Main Theorem IIIa for $\eta = \frac{1}{4}$ is based on [Rei14].

The relevant function spaces for the solutions are: $\mathrm{H}^1(\Omega; \mathbb{R}^{m_1})$ for $u^\varepsilon(t)$ and $u(t)$ as well as $\mathrm{H}^1(\Omega; \mathbb{R}^{m_2})$ for $v^\varepsilon(t)$ and $\mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y}; \mathbb{R}^{m_2}))$ for $V(t)$. Following the notation in Section 1.1, we abbreviate $\mathrm{H}^1(\Omega; \mathbb{R}^m)$ and $\mathrm{L}^2(\Omega; \mathbb{R}^m)$ for $m \in \mathbb{N}$ with $\mathrm{H}^1(\Omega)$ and $\mathrm{L}^2(\Omega)$, respectively. Throughout Chapter 2, these relevant spaces are denoted with

$$\begin{gathered} X = \mathrm{H}^1(\Omega), \;\; X_\varepsilon = (X, \|\cdot\|_{X_\varepsilon}), \text{ where } \|v\|_{X_\varepsilon} = \|v\|_H + \|\varepsilon \nabla v\|_H, \\ \mathbb{X} = \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y})), \;\; \mathbb{X}_0 = \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})), \\ H = \mathrm{L}^2(\Omega), \;\; \mathbb{H} = \mathrm{L}^2(\Omega \times \mathcal{Y}), \text{ and } \mathbb{H}_{\mathbb{R}^d} = \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y}), \end{gathered} \tag{2.0.6}$$

first introduced in (1.2.11), (1.2.12), and (1.2.14) in Section 1.2. It should be emphasized that the embedding $X \subset H$ is compact, whereas the two-scale space $\mathbb{X}$ embeds dense and continuously – but not compactly – into $\mathbb{H}$.

The structure of Chapter 2 is as follows. The Sections 2.1, 2.2 and 2.3 are devoted to the proof of Main Theorem I, II and IIIa–b, respectively. Finally, we summarize the chapter and give a brief outlook in Section 2.4.

2.1 Two-scale homogenization with improved time-regularity

This section is devoted to the rigorous proof of the limit passage from (2.0.1.P_ε) to (2.0.2.P_0) for solutions with improved time-regularity. The major difficulty is to prove the strong two-scale convergence of the slow diffusing variable v^ε, namely,

$$\begin{aligned} &\text{uniformly for all } t \in [0,T] : \; v^\varepsilon(t) \xrightarrow{2\mathrm{s}} V(t) \text{ in } \mathbb{H}, \\ &\text{i.e. } \max\nolimits_{0 \le t \le T} \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t) \|_{\mathbb{H}_{\mathbb{R}^d}} \to 0, \end{aligned} \tag{2.1.1}$$

which is not known a priori. (For the full version of Main Theorem I, we refer to page 33.) In [MRT14], which is the basis of this section, mainly the degenerating problem

$$v^\varepsilon_t = \mathrm{div}(\varepsilon^2 D^\varepsilon \nabla v^\varepsilon) + F^\varepsilon(v^\varepsilon) \quad \text{in } [0,T] \times \Omega \quad \text{with} \quad v^\varepsilon(0) = v^\varepsilon_0 \tag{2.1.2}$$

is considered. We expect v^ε to be bounded in the space X_ε, i.e. $\|v^\varepsilon\|_H + \|\varepsilon\nabla v^\varepsilon\|_H \le C$, see (2.1.10). Since X_ε is not compactly embedded into H uniformly for all $\varepsilon > 0$, the a priori boundedness of $(v^\varepsilon)_\varepsilon$ implies neither strong convergence in H nor strong two-scale convergence in $\mathbb{H}$ (up to subsequence).

The coupling of (2.1.2) to the classical u^ε-equations is rather easy, see [MRT14, Thm. 5.1], since the strong convergence $u^\varepsilon \to u$ in H is known a priori (at least up to subsequences). In view of the quantitative estimates in Section 2.3 as well as the strong convergence $\nabla u^\varepsilon \xrightarrow{2\mathrm{s}} \nabla u + \nabla_y U$ (which is not contained in [MRT14]), we apply our method developed for v^ε also to u^ε. Below, we sketch how to approach (2.1.1).

We will assume that $D^\varepsilon \rightsquigarrow \mathbb{D}$, $F^\varepsilon \rightsquigarrow \mathbb{F}$, and $v^\varepsilon \rightsquigarrow V_0$ in a suitable manner, specified in assumption (2.1.14.Conv) in Subsection 2.1.2. In view of the *a priori* weak convergences $v^\varepsilon \xrightarrow{2\mathrm{w}} V$ and $\varepsilon\nabla v^\varepsilon \xrightarrow{2\mathrm{w}} \nabla_y V$, see Theorem 1.2.5(b), we formally expect a result of the following type:

$$\begin{array}{ccc} \int_\Omega v_t^\varepsilon \cdot \varphi \,\mathrm{d}x & = & \int_\Omega -D^\varepsilon \varepsilon\nabla v^\varepsilon : \varepsilon\nabla\varphi + F^\varepsilon(v^\varepsilon)\cdot\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in X_\varepsilon \\ \downarrow & & \downarrow \quad \text{for } \varepsilon \to 0 \\ \int_{\Omega\times\mathcal{Y}} V_t \cdot \Phi \,\mathrm{d}x\,\mathrm{d}y & = & \int_{\Omega\times\mathcal{Y}} -D\nabla_y V : \nabla_y \Phi + F(V)\cdot\Phi \,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \Phi \in \mathbb{X}. \end{array} \tag{2.1.3}$$

To deduce the convergence of the weak forms (2.1.3), we have to cope with the fact that $(v^\varepsilon)_\varepsilon$ converges a priori only weakly in the two-scale sense and, therefore, the passage $F^\varepsilon(v^\varepsilon) \rightsquigarrow \mathbb{F}(V)$ is not straight forward, because F^ε and $\mathbb{F}$ are in general nonlinear. If we had the *strong* two-scale convergence of the sequence of solutions $(v^\varepsilon)_\varepsilon$, then $F^\varepsilon(v^\varepsilon) \rightsquigarrow \mathbb{F}(V)$ would follow easily. For the special case of F^ε being the gradient of a λ-convex potential ϕ, a rigorous convergence result of the type (2.1.3) was proved in [HJM94, Prop. 12] via methods of convex analysis and in particular Minty's Trick. In contrast to this, our approach to verify convergence (2.1.3), is to show that the sequence of solutions $(v^\varepsilon)_\varepsilon \subset \mathrm{L}^2(0,T;X_\varepsilon)$ converges even *strongly* in the two-scale sense to some limit $V \in \mathrm{L}^2(0,T;\mathbb{X})$, where $V \in \mathrm{H}^1(0,T;\mathbb{X})$ is the unique weak solution of the effective model

$$V_t = \mathrm{div}_y\,(\mathbb{D}\nabla_y V) + F(V) \quad \text{in } [0,T]\times\Omega\times\mathcal{Y} \quad \text{with} \quad V(0) = V_0. \tag{2.1.4}$$

The proof of convergence (2.1.3), in particular of the strong two-scale convergence $v^\varepsilon \xrightarrow{2\mathrm{s}} V$ in (2.1.1), relies on a clever choice of test functions suitable for the weak formulations of the ε– and the limit problem, i.e. (2.1.2) and (2.1.4), respectively. For the latter, suitable test functions must belong to $\mathbb{X}$, in particular they have to be Y-periodic. The most direct candidate $(\mathcal{T}_\varepsilon v^\varepsilon(t))_\varepsilon$ for $t \in [0,T]$ fixed supplies the required convergence but is *incompatible* with Y-periodicity, since $\mathcal{T}_\varepsilon v^\varepsilon(t) \in \widetilde{\mathbb{X}} = \mathrm{L}^2(\mathbb{R}^d, \mathrm{H}^1(Y))$, see *periodicity defect* (1.2.9). But the $\mathcal{T}_\varepsilon$-property (1.2.13) guarantees the *recovery of Y-periodicity for the limit*, which, thus, is *compatible* with the space of test functions of the limit problem. This is an essential observation for the proof of the strong two-scale convergence (2.1.1).

The improved time-regularity of the solutions v^ε and V via assumption (2.1.9.Time$_\varepsilon$) and (2.1.12.Time$_0$), respectively, is needed for the following technical reasons. In order to derive (2.0.5.Est), we need that $v_t^\varepsilon(t)$ and $V(t)$ are integrable functions (and not just elements of the dual spaces) so that we can apply the periodic unfolding operator $\mathcal{T}_\varepsilon$. Moreover, we apply Arzèla–Ascoli's theorem to the sequence $(v^\varepsilon)_\varepsilon$ to obtain a priori weak two-scale convergence uniformly in time. Finally, uniform bounds for $v_t^\varepsilon(t)$ are needed to control the folding mismatch $\Delta_1^{v^\varepsilon}$.

Section 2.1 is structured as follows. In Subsection 2.1.1, we begin with a review of existing literature concerning the theory of periodic homogenization for partial differentials equations in general and for systems involving slow diffusion in particular. In Subsection 2.1.2, we state Theorem 2.1.1 and all necessary assumptions. Based on these assumptions, we expound the existence of unique weak solutions $(u^\varepsilon, v^\varepsilon)$ of (2.0.1.P$_\varepsilon$) and (u, V) of (2.0.2.P$_0$), independently of the limit passage, as well as uniform a priori bounds. We briefly study the effective quantities D_{eff} and F_{eff} in Subsection 2.1.3. The abstract strategy of proving (2.1.1) is presented in Subsection 2.1.4. Subsection 2.1.5 is devoted to the derivation of (2.0.5.Est) and the error terms Δ^{u^ε} and Δ^{v^ε}. The control of those error terms as $\varepsilon \to 0$ and, hence, the proof of Theorem 2.1.1 as carried out in Subsection 2.1.6.

2.1.1 Review of existing literature on periodic homogenization

We briefly review some developments in periodic homogenization for elliptic and parabolic partial differential equations. For a detailed overview on homogenization theory of partial differential equations, we refer to the books [BLP78, JKO94, Hor97, Pan97, CiD99, MaK06, Tar09]. We begin with the classical linear elliptic equation

$$\operatorname{div}\big(\mathbb{D}(\tfrac{x}{\varepsilon})\nabla u^\varepsilon\big) = f \quad \text{in } \Omega \tag{2.1.5}$$

supplemented with homogeneous boundary conditions. We assume that $\mathbb{D}$ is uniformly elliptic, bounded, and Y-periodic as well as $f \in \mathrm{L}^2(\Omega)$. The a priori boundedness $\|u^\varepsilon\|_X \leq C$ implies the weak convergence $u^\varepsilon \rightharpoonup u$ in X (up to subsequences). Moreover, u solves the effective (homogenized) equation

$$\operatorname{div}\big(D_{\text{eff}}\nabla u\big) = f \quad \text{in } \Omega.$$

Here, the effective coefficients D_{eff} are indeed homogeneous (i.e. constant) and given via the standard unit cell problem (2.1.16). There are several methods to perform the limit passage $\varepsilon \to 0$ in (2.1.5) rigorously. For instance, G-convergence is exploited for a symmetric $\mathbb{D}$ in [Spa67, Spa68], it is done via compensated compactness in [Mur78, Tar79] for general $\mathbb{D}$, H-convergence is used in [MuT97], and two-scale convergence is applied in [Ngu89, All92]. With the same analytical techniques, we can treat the associated linear parabolic equation. As a second example, we consider the monotone parabolic equation

$$u_t^\varepsilon = \operatorname{div}\big(d^\varepsilon(t, x, \nabla u^\varepsilon)\big) + f \quad \text{in } [0,T] \times \Omega. \tag{2.1.6}$$

Based on two-scale convergence, rigorous homogenization results for (2.1.6) are proved in e.g. [FlO06, FH*07] including reiterated structures such as $d(t, x, x/\varepsilon, x/\varepsilon^2, \nabla u^\varepsilon)$, or in e.g. [HSW05, FlO07, Per12] including several temporal scales such as $d(t, t/\varepsilon^k, x, x/\varepsilon, \nabla u^\varepsilon)$, or in e.g. [ClP99, EK*10] including prefactors such as $c_\varepsilon(t,x)u_t^\varepsilon$, or in e.g. [NaR01] with $c_\varepsilon(t, x, u_t^\varepsilon)$.

While two-scale convergence is in general only meaningful for periodic settings, the theory of H-convergence is independent of any periodicity assumptions. However, for periodic microstructures, the method of two-scale convergence is much more powerful since it can easily be generalized to multiple spatial and temporal scales, it is suited for systems of equations, see e.g. [Tim13, Mah13]. Another advantage of two-scale convergence is that it applies to slow diffusion, whereas H-convergence is not suited for coefficients of the form $\varepsilon^2\mathbb{D}(x/\varepsilon)$ which degenerate as $\varepsilon \to 0$. Moreover, based on the periodic unfolding method, we have a very handy definition of strong two-scale convergence at hand which enables us to treat a bigger class of nonlinear problems. Although two-scale convergence is designed to treat periodic microstructures, it is also applicable if the coefficient functions are state-dependent (in a non-periodic fashion), see e.g. [PeB09, HaK15].

In the following, we review existing methods towards evolution processes with *slow diffusion*. The asymptotic behavior of *linear systems* of partial differential equations involving slow diffusion is considered in [PeB08, Pet09] with application to concrete carbonation, in [MiR13] for elastic waves in fluid-saturated porous media, and in [GrP14] for calcium dynamics in biological cells.

In [HJM94], systems of coupled reaction-diffusion equations in porous media are studied. Therein, the equations involving slow diffusion comprise reaction terms which are the *gradient of a convex potential* W, i.e. $F(v_\varepsilon) = \nabla_v W(v_\varepsilon)$. Based on the theory of *monotone operators* and Minty's lemma, the authors rigorously passed to the limit $\varepsilon \to 0$ relying only on the weak two-scale convergence. However, we emphasize that this approach does not apply to general reaction-diffusion systems since not all reaction terms are gradients.

The following four articles comprise nonlinear evolution problems and they define strong two-scale convergence via the periodic unfolding method (see Definition 1.2.3). The very different multiscale models on electromagnetism [Vis07], rate-independent systems [MiT07], plasticity [Han11], and ferromagnetism [Vis11] all have in common that the homogenization is based on *Γ-convergence* of *λ-convex functionals*. For this purpose, the construction of strongly two-scale converging recovery sequences is necessary in order to derive the Γ-$\limsup$ estimate. In this context, so-called *recovery operators* $\mathcal{G}_\varepsilon^\gamma$ are introduced in [Han11] for different scalings ε^γ of the gradient of solutions for $\gamma \in [0,\infty)$, cf. Definition 1.2.7 for $\gamma \in \{0,1\}$. These operators are used in our approach, too, in order to control the folding mismatch. Independently of the scaling ε^γ, a priori weak two-scale convergence of solutions is sufficient to identify the (two-scale) Γ-limit.

The following three articles consider the asymptotic behavior of reaction-diffusion systems involving slow diffusion and *nonlinear* reaction terms. All three rely on a notion of strong two-scale convergence based on (1.2.1). In [NeJ07], slow diffusion occurs only inside a ε-thin membrane, while classical diffusion is prevalent in the bulk of the domain. Using periodic boundary unfolding, introduced in [Neu96], the strong two-scale convergence of solutions inside the ε-thin membrane is deduced from a priori strong convergence in the bulk. For this purpose, the authors presume globally Lipschitz continuous nonlinearities as well as solutions with $\mathrm{L}^\infty(\Omega)$-bounds and improved time-regularity, i.e. $v_t^\varepsilon \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, such that Gronwall's lemma is applicable. A similar approach is used in [PtR10] to derive a macroscopic model for transport of strongly sorbed solute in the soil. Therein, the reaction-diffusion processes are set in a porous medium and slow diffusion occurs only inside the pores. Again, the strong two-scale convergence inside the pores is deduced from a priori strong convergence outside the pores. In [GPS14], a coupled system with classical diffusion only and ε^m-scaled reactions on the pore surface are investigated for $m \in \mathbb{N}$. We point out that all presented methods yield rigorous homogenization results.

In [Eck05, Muv13], the asymptotic behavior of reaction-diffusion systems involving slow diffusion and nonlinear reactions is considered as well. Based on the formal method of *asymptotic expansion* and *Gronwall-type estimates*, effective model equations are derived. Moreover, the authors prove quantitative estimates for the original solutions and the macroscopic reconstruction of the effective solutions. We refer to Subsection 2.3.9 for a comparison of their convergence rates with ours. Formal asymptotic expansion is also employed in [FA*11] to study a semilinear parabolic system in a locally periodic perforated domain with application to sulfate attack in sewer pipes.

2.1.2 Assumptions and a priori bounds for solutions

We write (2.0.1.P$_\varepsilon$) shortly in the form

$$\begin{aligned} w^\varepsilon_t &= \operatorname{div}(D^\varepsilon \nabla w^\varepsilon) + F^\varepsilon(w^\varepsilon) && \text{in } [0,T]\times\Omega, \\ w^\varepsilon(0) &= w^\varepsilon_0 && \text{in } \Omega, \end{aligned} \tag{2.1.7.P$_\varepsilon$}$$

provided with homogeneous Neumann boundary conditions on $\partial\Omega$. We are looking for solutions $w^\varepsilon := (u^\varepsilon, v^\varepsilon) : [0,T]\times\Omega \to \mathbb{R}^{m_1+m_2}$ of (2.1.7.P$_\varepsilon$) with $w^\varepsilon_0 = (u^\varepsilon_0, v^\varepsilon_0)$. The diffusion tensor $D^\varepsilon : [0,T]\times\Omega \to \mathbb{R}^{([m_1+m_2]\times d)\times([m_1+m_2]\times d)}$ and the reaction term $F^\varepsilon : [0,T]\times\Omega\times\mathbb{R}^{m_1+m_2} \to \mathbb{R}^{m_1+m_2}$ are of the form

$$D^\varepsilon = \begin{pmatrix} D^\varepsilon_1 & 0 \\ 0 & \varepsilon^2 D^\varepsilon_2 \end{pmatrix} \quad \text{and} \quad F^\varepsilon(w) = \begin{pmatrix} F^\varepsilon_2(u,v) \\ F^\varepsilon_1(u,v) \end{pmatrix}. \tag{2.1.8}$$

Here, and throughout this section, we postulate the following assumptions on the given data of (2.1.7.P$_\varepsilon$).

There exists $C_0 \geq 0$ such that for all $\varepsilon > 0$:

Existence: For $i \in \{1,2\}$,
$$D^\varepsilon_i \in \mathcal{M}(\Omega),\ F^\varepsilon_i \in \mathcal{F}(\Omega),\ \text{and } \|u^\varepsilon_0\|_H + \|v^\varepsilon_0\|_H \leq C_0; \tag{2.1.9.Exist$_\varepsilon$}$$

Improved time-regularity:
$$\|\operatorname{div}(D^\varepsilon_1(0)\nabla u^\varepsilon_0)\|_H + \|\operatorname{div}(\varepsilon^2 D^\varepsilon_2(0)\nabla v^\varepsilon_0)\|_H \leq C_0. \tag{2.1.9.Time$_\varepsilon$}$$

We recall that the diffusion tensors D^ε_i are uniformly elliptic and bounded and the reaction terms F^ε_i are globally Lipschitz continuous, see (1.1.7).

Since for $\varepsilon > 0$ fixed, both norms, $\|\cdot\|_X$ and $\|\cdot\|_{X_\varepsilon}$, are equivalent, we have that $X = X_\varepsilon \subset H$. Thus $X_\varepsilon \subset H$ is dense and continuously embedded and we obtain that $X \times X_\varepsilon \subset H \times H \subset X^* \times X^*_\varepsilon$ is an evolution triple. For given $T > 0$, the assumptions (2.1.9.Exist$_\varepsilon$) and (2.1.9.Time$_\varepsilon$) imply according to Theorem 1.1.2 and Proposition 1.1.3 the existence of a unique solution w^ε of (2.1.7.P$_\varepsilon$) with $u^\varepsilon \in W_{\mathrm{imp}}(0,T;X)$ and $v^\varepsilon \in W_{\mathrm{imp}}(0,T;X_\varepsilon)$, see (1.1.3) and (1.1.13) for the definition of the spaces. In particular, the solutions satisfy the uniform bounds

$$\begin{aligned} \|u^\varepsilon\|_{C^1([0,T];H)} + \|u^\varepsilon\|_{\mathrm{H}^1(0,T;X)} + \|u^\varepsilon\|_{\mathrm{H}^2(0,T;X^*)} &\leq C_b, \\ \|v^\varepsilon\|_{C^1([0,T];H)} + \|v^\varepsilon\|_{\mathrm{H}^1(0,T;X_\varepsilon)} + \|v^\varepsilon\|_{\mathrm{H}^2(0,T;X^*_\varepsilon)} &\leq C_b. \end{aligned} \tag{2.1.10}$$

To determine the constant C_b, we have used that, by (2.1.9.Exist$_\varepsilon$), the diffusion tensors D_i^ε, respective reaction terms F_i^ε, belong to the same class for all $\varepsilon > 0$, namely $\mathcal{M}(\Omega)$ respective $\mathcal{F}(\Omega)$. Therefore, $\varepsilon^2\beta/\sqrt{\varepsilon^2\alpha} \leq \beta/\sqrt{\alpha}$ and $\|\varepsilon^2 D_i^\varepsilon\|_{C^1([0,T];\mathrm{L}^\infty(\Omega))}/(\varepsilon^2\alpha) \sim O(1)$ provide a uniform bound for v^ε in Theorem 1.1.2 and Proposition 1.1.3 for all $\varepsilon \in (0,1]$. In view of the definition of X_ε, we obtain the existence of a constant $C_b \geq 0$, independent of ε, such that (2.1.10) holds for all $\varepsilon \in (0,1]$.

In the same manner, we reformulate the limit system (2.0.2.P$_0$) and show the existence of solutions. For $W(t,x,y) := (u(t,x), V(t,x,y)) : [0,T] \times \Omega \times \mathcal{Y} \to \mathbb{R}^{m_1+m_2}$, we write

$$\begin{aligned} W_t &= \begin{pmatrix} \mathrm{div}(D_{\mathrm{eff}}\nabla u) \\ \mathrm{div}_y(\mathbb{D}_2\nabla_y V) \end{pmatrix} + \begin{pmatrix} F_{\mathrm{eff}}(W) \\ \mathbb{F}_2(W) \end{pmatrix} && \text{in } [0,T]\times\Omega\times\mathcal{Y}, \\ W(0) &= W_0 && \text{in } \Omega\times\mathcal{Y}, \end{aligned} \tag{2.1.11.P$_0$}$$

provided with homogeneous Neumann boundary conditions for u on $\partial\Omega$ and, by definition of $\mathcal{Y}$, periodic boundary conditions for V. Due to insufficient regularity, boundary conditions for V on $\partial\Omega$ are not meaningful. The effective diffusion tensor D_{eff} is obtained from the given two-scale tensor $\mathbb{D}_1$ via solving the unit cell problem, see (2.1.16)–(2.1.17). Similarly, averaging the two-scale reaction term $\mathbb{F}_1$ over $\mathcal{Y}$ yields the effective reaction F_{eff}, see (2.0.3). Throughout this section, the following assumptions on the given data of (2.1.11.P$_0$) hold true

$$\begin{aligned} &\textit{Existence:}\ \text{ For } i \in \{1,2\}, \\ &\mathbb{D}_i \in \mathcal{M}(\Omega\times\mathcal{Y}),\ \mathbb{F}_i \in \mathcal{F}(\Omega\times\mathcal{Y}),\ \text{ and } u_0 \in H, V_0 \in \mathbb{H}; \end{aligned} \tag{2.1.12.Exist$_0$}$$

$$\begin{aligned} &\textit{Improved time-regularity:} \\ &\mathrm{div}(D_{\mathrm{eff}}(0)\nabla u_0) \in H \text{ and } \mathrm{div}_y(\mathbb{D}_2(0)\nabla_y V_0) \in \mathbb{H}. \end{aligned} \tag{2.1.12.Time$_0$}$$

It is easy to check that $F_{\mathrm{eff}} \in \mathcal{F}(\Omega)$ with the same parameters as $\mathbb{F}$ and D_{eff} again satisfies $D_{\mathrm{eff}} \in \mathcal{M}(\Omega)$ with $\alpha_{\mathrm{eff}} = \alpha$ and $\beta_{\mathrm{eff}} = \beta^2/\alpha$, see e.g. [CiD99, Thm. 13.4] or [MuT97, Thm. 2].

We point out that the two-scale spaces $\mathbb{X}$ and $\mathbb{H}$ generate the evolution triple $X \times \mathbb{X} \subset H \times \mathbb{H} \subset X^* \times \mathbb{X}^*$. For given $T > 0$ and initial value $W_0 = (u_0, V_0)$, the assumptions (2.1.12.Exist$_0$)–(2.1.12.Time$_0$) imply according to Theorem 1.1.2 and Proposition 1.1.3 the existence of a unique solution W of (2.1.11.P$_0$) with $u \in W_{\mathrm{imp}}(0,T;X)$ and $V \in W_{\mathrm{imp}}(0,T;\mathbb{X})$. Moreover, we obtain the boundedness

$$\begin{aligned} \|u\|_{C^1([0,T];H)} + \|u\|_{\mathrm{H}^1(0,T;X)} + \|u\|_{\mathrm{H}^2(0,T;X^*)} &\leq C_b, \\ \|V\|_{C^1([0,T];\mathbb{H})} + \|V\|_{\mathrm{H}^1(0,T;\mathbb{X})} + \|V\|_{\mathrm{H}^2(0,T;\mathbb{X}^*)} &\leq C_b, \end{aligned} \tag{2.1.13}$$

where the constant C_b depends on the same quantities as in (2.1.10).

Finally, we assume that the given data of (2.1.7.P$_\varepsilon$) in (2.1.9.Exist$_\varepsilon$) converge in the following sense to the data of (2.1.11.P$_0$) in (2.1.12.Exist$_0$).

$$\begin{aligned} &\textit{Convergence of the given data for } i \in \{1,2\}: \\ &\mathcal{T}_\varepsilon D_i^\varepsilon(t,x,y) \to \mathbb{D}_i^{\mathrm{ex}}(t,x,y) \text{ for a.a. } (x,y) \in \Omega\times\mathcal{Y} \text{ and all } t \in [0,T], \\ &F_i^\varepsilon(t,\cdot,A,B) \to \mathbb{F}_i^{\mathrm{ex}}(t,\cdot,\cdot,A,B) \text{ in } \mathbb{H} \text{ for all } (t,A,B) \in [0,T]\times\mathbb{R}^{m_1+m_2}, \\ &\text{as well as } u_0^\varepsilon \to u_0 \text{ in } H \text{ and } v_0^\varepsilon \xrightarrow{2\mathrm{s}} V_0 \text{ in } \mathbb{H}. \end{aligned} \tag{2.1.14.Conv}$$

Having collected all assumptions, we can now state the full version of Main Theorem I.

Theorem 2.1.1 (Main Theorem I). *Let the assumptions* (2.1.9.Exist$_\varepsilon$), (2.1.9.Time$_\varepsilon$), (2.1.12.Exist$_0$), (2.1.12.Time$_0$), *and* (2.1.14.Conv) *be satisfied. Then, the sequence of solutions* $(u^\varepsilon, v^\varepsilon)_\varepsilon$ *of* (2.0.1.P$_\varepsilon$) *converges to the solution* (u, V) *of* (2.0.2.P$_0$) *in the following sense*

$$\begin{aligned}&\textit{uniformly for all } t \in [0,T]:\ v^\varepsilon(t) \xrightarrow{2\mathrm{s}} V(t)\ \textit{in } \mathbb{H},\\ &\textit{i.e.}\ \max_{0\le t\le T} \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} \to 0,\end{aligned} \tag{2.1.15a}$$

$$\begin{aligned}&\textit{pointwise for all } t \in [0,T]:\ \varepsilon\nabla v^\varepsilon(t) \xrightarrow{2\mathrm{s}} \nabla_y V(t)\ \textit{in } \mathbb{H}\\ &\textit{and } \varepsilon\nabla v^\varepsilon \xrightarrow{2\mathrm{s}} \nabla_y V\ \textit{in } \mathrm{L}^2(0,T;\mathbb{H}),\end{aligned} \tag{2.1.15b}$$

$$v^\varepsilon_t \xrightarrow{2\mathrm{w}} V_t\ \textit{in } \mathrm{L}^2(0,T;\mathbb{H}), \tag{2.1.15c}$$

$$\begin{aligned}&u^\varepsilon \rightharpoonup u\ \textit{in } \mathrm{L}^2(0,T;X)\ \textit{and } u^\varepsilon_t \rightharpoonup u_t\ \textit{in } \mathrm{L}^2(0,T;X^*),\ \textit{moreover}\\ &\textit{there exists a function } U \in \mathrm{L}^2(0,T;\mathbb{X}_0)\ \textit{such that}\\ &\textit{pointwise for all } t \in [0,T]:\ \nabla u^\varepsilon(t) \xrightarrow{2\mathrm{s}} \nabla u(t) + \nabla_y U(t)\ \textit{in } \mathbb{H}.\end{aligned} \tag{2.1.15d}$$

In (2.1.15d)$_3$, the one-scale function u is understood as two-scale function via the canonical embedding E, cf. (1.2.10). For notational simplicity, we omit E throughout this chapter. In Theorem 2.1.1, we show, additionally to [MRT14, Thm. 4.1], the strong two-scale convergence of the gradients ∇u^ε in (2.1.15d). Therefore, we apply the same calculations that we apply to v^ε as well to u^ε. Moreover, the quantitative estimates in Section 2.3 rely on this section, in particular on (2.0.5.Est).

Remark 2.1.2 (Evolutionary Γ-convergence for gradient structures). *We briefly sketch an alternative approach to [HJM94] based on Minty's Trick, if the problem* (2.1.2) *under consideration admits a gradient flow structure.*

For simplicity, let the solution v^ε *of* (2.1.2) *be scalar-valued. If the matrix* D^ε *is symmetric and* F^ε *is the derivative of a convex potential* W*, i.e.* $D^\varepsilon(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *and* $F^\varepsilon(x,v) \equiv \partial_v W(v)$*, then equation* (2.1.2) *admits a gradient structure (cf. Section 3.1). One possible gradient structure* $(X_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *is the following: On the state space* $X_\varepsilon = \mathrm{L}^2(\Omega)$*, we consider the energy functional and the dissipation potential*

$$\mathcal{E}_\varepsilon(v) = \textstyle\int_\Omega \frac{1}{2}\nabla v \cdot \varepsilon^2 D^\varepsilon(x)\nabla v - W(v)\,\mathrm{d}x \quad \textit{and} \quad \mathcal{R}_\varepsilon(v_t) = \textstyle\int_\Omega \frac{1}{2}|v_t|^2\,\mathrm{d}x,$$

respectively. With this, (2.1.2) *is equivalent to the force-balance relation* $0 = \mathrm{D}\mathcal{R}_\varepsilon(v^\varepsilon) + \mathrm{D}\mathcal{E}_\varepsilon(v^\varepsilon)$*. Following [Han11], we obtain that* $\mathcal{E}_\varepsilon$ *and* $\mathcal{R}_\varepsilon$ *weakly* Γ*-converge in the two-scale sense to* $\mathcal{E}_0$ *and* $\mathcal{R}_0$*, respectively, where the two-scale* Γ*-limits are given via*

$$\mathcal{E}_0(V) = \textstyle\int_{\Omega\times\mathcal{Y}} \frac{1}{2}\nabla_y V \cdot \mathbb{D}(x,y)\nabla_y V - W(V)\,\mathrm{d}x\,\mathrm{d}y \quad \textit{and} \quad \mathcal{R}_0(V_t) = \textstyle\int_{\Omega\times\mathcal{Y}} \frac{1}{2}|V_t|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Based on the energy-dissipation principle (cf. Subsection 3.1.3), one may eventually prove that $(X_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges *to* $(X_0, \mathcal{E}_0, \mathcal{R}_0)$ *with* $X_0 = \mathrm{L}^2(\Omega\times\mathcal{Y})$*, if the initial values are well-prepared (cf. Definition 3.1.2). In this sense, we may call the additional assumptions* (2.1.9.Time$_\varepsilon$) *and* (2.1.12.Time$_0$) *for our reaction-diffusion systems* well-preparedness of the initial conditions *as in the context of E-convergence.*

2.1.3 On the unit cell problem (for classical diffusion)

We briefly recall well-known facts concerning the effective diffusion tensor D_{eff} in the u-equations $(2.0.2.\mathrm{P}_0)_1$. The effective diffusion tensor $D_{\text{eff}} : [0,T] \times \Omega \to \mathbb{R}^{(m_1\times d)\times(m_1\times d)}$ is given component wise via the standard homogenization formula, see e.g. [BLP78, All92, LNW02],

$$D_{\text{eff}}(t,x)_{ijkl} := \fint_{\mathcal{Y}} \mathbb{D}_1(t,x,y)_{ijkl} + \sum_{r=1}^{d} \mathbb{D}_1(x,y)_{ijkr} \cdot \partial_{y_r} z(t,x,y)_{kl}\,\mathrm{d}y, \tag{2.1.16}$$

for $i,k = 1,\dots,m_1$, $j,l = 1,\dots,d$. Here, the so-called correctors $z_{ij} : [0,T]\times\Omega\times\mathcal{Y} \to \mathbb{R}^d$ solve for a.e. $(t,x) \in [0,T]\times\Omega$ the local problem in the weak sense:

$$\operatorname{div}_y \left(\mathbb{D}_1(t,x,y)_{ijkl} + \sum_{r=1}^{d} \mathbb{D}_1(t,x,y)_{ijkr} \cdot \partial_{y_r} z(t,x,y)_{kl} \right) = 0 \quad \text{in } \mathcal{Y}. \tag{2.1.17}$$

We study existence and regularity of the corrector functions in Lemma 2.1.5, below.

Remark 2.1.3. *In equation* (2.1.16)*, we denote with* $\fint_{\mathcal{Y}} \square\,\mathrm{d}y := (\mathrm{vol}(\mathcal{Y}))^{-1} \int_{\mathcal{Y}} \square\,\mathrm{d}y$ *the man value with* vol *denoting the d-dimensional Lebesgue measure. Since the torus* $\mathcal{Y}$ *is a d-dimensional hypersurface in* $\mathbb{R}^{d+1}$*, the volume* $\mathrm{vol}(\mathcal{Y})$ *actually describes the d-dimensional surface area. In this sense, we have by construction* $\mathrm{vol}(Y) = 1$ *as well as* $\mathrm{vol}(\mathcal{Y}) = 1$.

If $\mathbb{D}_1$ is symmetric in the sense $\mathbb{D}_1\xi_1 : \xi_2 = \xi_1 : \mathbb{D}_1\xi_2$ for all $\xi_1, \xi_2 \in \mathbb{R}^{m_1\times d}$, then D_{eff} is represented via the minimization problem, often referred to as unit cell problem,

$$D_{\text{eff}}(t,x)\xi : \xi = \min_{\varphi \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})} \fint_{\mathcal{Y}} \mathbb{D}_1(t,x,y)(\nabla_y\varphi(y) + \xi) : (\nabla_y\varphi(y) + \xi)\,\mathrm{d}y. \tag{2.1.18}$$

In the one-dimensional case, the effective diffusion coefficient is explicitly given by the harmonic mean $D_{\text{eff}}(t,x) = (\int_0^1 (\mathbb{D}_1(t,x,y))^{-1}\,\mathrm{d}y)^{-1}$. The unique minimizer φ_ξ of (2.1.18) solves the associated Euler-Lagrange equation $\operatorname{div}_y(\mathbb{D}_1[\xi + \nabla_y\varphi_\xi]) = 0$ in $\mathcal{Y}$. For ξ being a canonical basis matrix in $\mathbb{R}^{m_1\times d}$, the corrector function φ_ξ is identical to z_{ij} in (2.1.17).

We now provide an equivalent characterization of (2.1.16)–(2.1.17) in the weak sense. In this section, we consider an arbitrarily given function u and then, throughout the remainder of this chapter, u denotes the effective solution in the case of classical diffusion.

Lemma 2.1.4. *Let* $u \in \mathrm{L}^2(0,T;X)$ *be given. Then, the effective diffusion tensor* D_{eff} *in* (2.1.16) *satisfies for all* $(\psi,\Psi) \in X \times \mathbb{X}_0$

$$\int_\Omega D_{\text{eff}}\nabla u : \nabla\psi\,\mathrm{d}x = \int_{\Omega\times\mathcal{Y}} \mathbb{D}_1\left(\nabla u + \nabla_y U\right) : \left(\nabla\psi + \nabla_y\Psi\right)\mathrm{d}x\,\mathrm{d}y. \tag{2.1.19}$$

Here, U *denotes the corrector function corresponding to* u*, namely*

$$U(t,x,y)_i = \sum_{j=1}^{d} \frac{\partial u_i}{\partial x_j}(t,x) \cdot z(t,x,y)_{ij} \tag{2.1.20}$$

for $i = 1.,\dots,m_1$ *and* z_{ij} *solves* (2.1.17).

Proof. We keep $t \in [0,T]$ fixed and suppress it in the notation. Without loss of generality, we set $m_1 = 1$ and we denote with $\{e_j\}_j$ the canonical orthonormal basis in $\mathbb{R}^d$. Thus, (2.1.16)–(2.1.17) reads for each $j \in \{1,\dots,d\}$

$$D_{\text{eff}}(x)e_j = \fint_{\mathcal{Y}} \mathbb{D}_1(x,y)[e_j + \nabla_y z(x,y)_j]\,\mathrm{d}y \quad \text{with} \quad -\operatorname{div}_y\left(\mathbb{D}_1(x,y)[e_j + \nabla_y z_j]\right) = 0.$$

Since D_{eff} acts linearly on the vector which it corrects, we obtain for the given vector $\nabla u(x) \in \mathbb{R}^d$

$$D_{\text{eff}}(x)\nabla u(x) = \fint_{\mathcal{Y}} \mathbb{D}_1(x,y)\left[\nabla u(x) + \nabla_y\left(\sum_{j=1}^d \frac{\partial u}{\partial x_j}(x)\cdot z(x,y)_j\right)\right]\mathrm{d}y.$$

Defining the function U as in (2.1.20), we obtain for a.e. $x \in \Omega$

$$D_{\text{eff}}\nabla u = \fint_{\mathcal{Y}} \mathbb{D}_1[\nabla u + \nabla_y U]\,\mathrm{d}y \quad \text{with} \quad -\operatorname{div}_y\left(\mathbb{D}_1[\nabla u + \nabla_y U]\right) = 0. \tag{2.1.21}$$

Testing $(2.1.21)_1$ with an arbitrary function $\psi \in X$ gives (with $\operatorname{vol}(\mathcal{Y}) = 1$)

$$\int_\Omega D_{\text{eff}}\nabla u : \nabla\psi\,\mathrm{d}x = \int_{\Omega\times\mathcal{Y}} \mathbb{D}_1[\nabla u + \nabla_y U] : \nabla\psi\,\mathrm{d}x\,\mathrm{d}y. \tag{2.1.22}$$

The weak formulation of $(2.1.21)_2$ reads

$$\int_{\Omega\times\mathcal{Y}} \mathbb{D}_1[\nabla u + \nabla_y U] : \nabla_y\Psi\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{for all } \Psi \in \mathbb{X}_0.$$

Adding 0 to the right-hand side of (2.1.22) and using the latter equality yields (2.1.19). □

In what follows, we prove the existence of the correctors z_{ij} as solutions of (2.1.17) and we comment on the regularity of U in (2.1.20).

Lemma 2.1.5. *Let $\mathbb{D}_1 \in \mathcal{M}(\Omega\times\mathcal{Y})$ and $i = 1,\dots,m_1$, $j = 1,\dots,d$.*

(a) There exists a unique solution $z_{ij} \in W^{1,\infty}(0,T;\mathrm{L}^\infty(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$ of the local problem (2.1.17).

(b) If we have additionally $\partial_{x_j}\mathbb{D}_1 \in C^1([0,T];\mathrm{L}^\infty(\Omega\times\mathcal{Y}))$, then the correctors satisfy the higher regularity $z_{ij} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$.

Depending on the regularity of $\mathbb{D}_1$ and ∇u, we obtain the following regularity for U:

$$\begin{aligned}
&\mathbb{D}_1 \in \mathrm{L}^\infty(\Omega\times\mathcal{Y}) \text{ and } \nabla u \in \mathrm{L}^2(0,T;H): && U \in \mathrm{L}^2(0,T;\mathbb{X}_0)\\
&\partial_t\mathbb{D}_1 \in \mathrm{L}^\infty(\Omega\times\mathcal{Y}) \text{ and } \nabla u \in C([0,T];H): && U \in C([0,T];\mathbb{X}_0)\\
&\partial_{x_j}\mathbb{D}_1 \in \mathrm{L}^\infty(\Omega\times\mathcal{Y}) \text{ and } \nabla u \in \mathrm{L}^2(0,T;X): && U \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))).
\end{aligned} \tag{2.1.23}$$

Proof of Lemma 2.1.5. *Ad (a).* Without loss of generality, we set $m_1 = 1$. We seek solutions $z_j : [0,T]\times\Omega\times\mathcal{Y} \to \mathbb{R}$ of (2.1.17), for $j = 1,\dots,d$, namely

$$\operatorname{div}_y\left(\mathbb{D}_1[e_j + \nabla_y z_j]\right) = 0 \quad \text{in } \mathcal{Y}, \tag{2.1.24}$$

where $\{e_j\}_j$ denotes the canonical orthonormal basis in $\mathbb{R}^d$. The diffusion tensor $\mathbb{D}_1$ defines a coercive and bounded bilinear form $\mathbb{B}$ on the Hilbert space $\mathrm{L}^2(0,T;\mathbb{X}_0)$ by setting $\mathbb{B}(v,w) := \int_0^T\int_{\Omega\times\mathcal{Y}} \mathbb{D}_1\nabla_y v : \nabla_y w\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$. Hence the Lax–Milgram theorem yields for every j the existence of a unique $z_j \in \mathrm{L}^2(0,T;\mathbb{X}_0)$ satisfying (2.1.24). Moreover, we obtain $z_j \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$ as follows: we integrate (2.1.24) over $\mathcal{Y}$, test with z_j, and use $\mathrm{vol}(\mathcal{Y}) = 1$ such that

$$\sup_{(t,x)\in[0,T]\times\Omega} \|z(t,x,\cdot)_j\|_{\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})} \leq \frac{\beta}{\alpha}. \tag{2.1.25}$$

We continue as in Proposition 1.1.3 by setting $\widetilde{z}_j := \partial_t z_j$ and differentiating (2.1.24) w.r.t. time, i.e. $\mathrm{div}_y\left(\partial_t \mathbb{D}_1[e_j + \nabla_y z_j] + \mathbb{D}_1 \nabla_y \widetilde{z}_j\right) = 0$. Using $\partial_t \mathbb{D}_1 \in C([0,T];\mathrm{L}^\infty(\Omega\times\mathcal{Y}))$, we obtain once more by Lax–Milgram's theorem $\widetilde{z}_j \in \mathrm{L}^2(0,T;\mathbb{X}_0)$ and by (2.1.25) $\widetilde{z}_j \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$. Overall, we arrive at $z_j \in W^{1,\infty}(0,T;\mathrm{L}^\infty(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$.

Ad (b). Proceeding as before, we differentiate (2.1.24) w.r.t. x_i, for $i = 1,\ldots,d$, and we obtain a solution $z \in \mathrm{L}^\infty(0,T;W^{1,\infty}(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$ due to $\partial_{x_i}\mathbb{D}_1 \in C^1([0,T];\mathrm{L}^\infty(\Omega\times\mathcal{Y}))$. Finally, differentiating one more time w.r.t. time t yields $z \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$. To rigorously justify the formal differentiation w.r.t. x_i, we can argue as in Step 1 in the proof of Proposition 2.3.17. □

2.1.4 Abstract strategy for proving strong two-scale convergence

To highlight the general approach to the proof of the strong two-scale convergence result (2.1.15a), we consider the two abstract systems

$$v^\varepsilon_t = \mathcal{A}^\varepsilon v^\varepsilon + F^\varepsilon(v^\varepsilon) \quad \text{and} \quad V_t = \mathbb{A}V + \mathbb{F}(V) \tag{2.1.26}$$

in the Hilbert spaces $\mathcal{X} \subset \mathcal{H}$ and $\mathbb{X} \subset \mathbb{H}$, respectively. The operators $\mathcal{A}^\varepsilon$ and $\mathbb{A}$ are given in terms of uniformly bounded and uniformly elliptic quadratic forms, namely

$$\mathcal{B}_\varepsilon(v,w) = \langle -\mathcal{A}^\varepsilon v, w\rangle \quad \text{and} \quad \mathbb{B}(V,W) = \langle\!\langle -\mathbb{A}V, W\rangle\!\rangle.$$

We consider an unfolding operator $\mathcal{T}_\varepsilon : \mathcal{H} \to \mathbb{H}$ which also satisfies $\mathcal{T}_\varepsilon : \mathcal{X} \to \widetilde{\mathbb{X}}$, where $\mathbb{X} \subsetneq \widetilde{\mathbb{X}}$ is a closed subspace. For the corresponding folding operators $\mathcal{F}_\varepsilon : \mathbb{H} \to \mathcal{H}$ and $\mathcal{G}_\varepsilon : \mathbb{X} \to \mathcal{X}$, we assume that $\mathcal{T}_\varepsilon{}' = \mathcal{F}_\varepsilon$ and that $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$ are comparable in the sense of Proposition 1.2.9.

We want to show that the solution v^ε converges to V, i.e. $W^\varepsilon \to 0$ in $\mathbb{H}$ or $w^\varepsilon \to 0$ in $\mathcal{H}$, where

$$W^\varepsilon := \mathcal{T}_\varepsilon v^\varepsilon - V \quad \text{and} \quad w^\varepsilon := v^\varepsilon - \mathcal{G}_\varepsilon V.$$

For the proof we resort to working with W^ε instead of w^ε, since this gives the desired two-scale convergence more directly. In particular, to establish this convergence for $(W^\varepsilon)_\varepsilon$, we derive a Gronwall estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2 \leq L\|W^\varepsilon\|^2 + \Delta^\varepsilon, \tag{2.1.27}$$

where $\|\cdot\|$ stands for the norm in the Hilbert space $\mathbb{H}$. From

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2 = \langle\!\langle W^\varepsilon_t, W^\varepsilon\rangle\!\rangle = \langle\!\langle (\mathcal{T}_\varepsilon v^\varepsilon)_t, W^\varepsilon\rangle\!\rangle - \langle\!\langle V_t, W^\varepsilon\rangle\!\rangle \tag{2.1.28}$$

we see that it is desirable to test the equations (2.1.26) with $\mathcal{F}_\varepsilon W^\varepsilon$ and W^ε, respectively. However this is not possible as we do neither have $\mathcal{F}_\varepsilon W^\varepsilon \in \mathcal{X}$ nor $W^\varepsilon \in \mathbb{X}$. Indeed $w^\varepsilon \in \mathcal{X}$ is an admissible test function for $(2.1.26)_1$, but $\mathcal{T}_\varepsilon w^\varepsilon \notin \mathbb{X}$, due to $\mathcal{T}_\varepsilon v^\varepsilon \in \widetilde{\mathbb{X}} \supsetneqq \mathbb{X}$.

Observe that $\mathcal{B}_\varepsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, whereas $\mathbb{B} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. To overcome this discrepancy in the underlying spaces $\mathcal{X}$ and $\mathbb{X}$, we replace $\mathcal{B}_\varepsilon$ with a quadratic form $\mathbb{B}_\varepsilon : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ with the same properties as $\mathcal{B}_\varepsilon$ and compensate their mismatch by an additional error term. Thus, we obtain four different types of errors, namely

1. Δ_1^ε for the *folding mismatch* between $\mathcal{F}_\varepsilon V$ and $\mathcal{G}_\varepsilon V$,
2. Δ_2^ε for the *incompatibility* of $\mathcal{T}_\varepsilon v^\varepsilon \in \widetilde{\mathbb{X}} \supsetneqq \mathbb{X}$,
3. Δ_3^ε for the *approximation error* between $\mathbb{B}$ and $\mathbb{B}_\varepsilon$, and
4. Δ_4^ε for the *approximation error* between F^ε and $\mathbb{F}$.

More precisely, we test $(2.1.26)_1$ with $w^\varepsilon = (v^\varepsilon - \mathcal{F}_\varepsilon V) + (\mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon V)$, transform the equation from $\mathcal{X}$ to $\mathbb{X}$ using $\mathcal{T}_\varepsilon$ and $\mathbb{B}_\varepsilon$ so that we obtain

$$\langle\!\langle (\mathcal{T}_\varepsilon v^\varepsilon)_t, W^\varepsilon \rangle\!\rangle = -\mathbb{B}_\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon, W^\varepsilon) + \langle\!\langle \mathcal{T}_\varepsilon F^\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon), W^\varepsilon \rangle\!\rangle + \Delta_1^\varepsilon, \tag{2.1.29}$$

$$\begin{aligned} \text{where } \Delta_1^\varepsilon &:= \langle\!\langle (\mathcal{T}_\varepsilon v^\varepsilon)_t, W^\varepsilon \rangle\!\rangle + \mathbb{B}_\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon, W^\varepsilon) - \langle\!\langle \mathcal{T}_\varepsilon F^\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon), W^\varepsilon \rangle\!\rangle \\ &\quad - \langle v_t^\varepsilon, w^\varepsilon \rangle \quad - \quad \mathcal{B}_\varepsilon(v^\varepsilon, w^\varepsilon) \quad + \quad \langle F^\varepsilon(v^\varepsilon), w^\varepsilon \rangle. \end{aligned}$$

We may additionally assume that $\mathbb{B}$ is well-defined on $\widetilde{\mathbb{X}}$ as well. However, testing $(2.1.26)_2$ with $W^\varepsilon \in \widetilde{\mathbb{X}}$ is not allowed, since equation $(2.1.26)_2$ is valid in the subspace $\mathbb{X}$, only. Nevertheless, each of the expressions $\langle\!\langle V_t, W^\varepsilon \rangle\!\rangle$, $\mathbb{B}(V, W^\varepsilon)$, $\langle\!\langle \mathbb{F}(V), W^\varepsilon \rangle\!\rangle$ is well-defined. Therefore we test $(2.1.26)_2$ with V only, include the missing terms containing $\mathcal{T}_\varepsilon v^\varepsilon$ and compensate them by the *periodicity defect error term* Δ_2^ε via

$$\begin{aligned} \langle\!\langle V_t, W^\varepsilon \rangle\!\rangle &= -\langle\!\langle V_t, V \rangle\!\rangle + \langle\!\langle V_t, \mathcal{T}_\varepsilon v^\varepsilon \rangle\!\rangle = \mathbb{B}(V,V) - \langle\!\langle \mathbb{F}(V), V \rangle\!\rangle + \langle\!\langle V_t, \mathcal{T}_\varepsilon v^\varepsilon \rangle\!\rangle \\ &= -\mathbb{B}(V, W^\varepsilon) + \langle\!\langle \mathbb{F}(V), W^\varepsilon \rangle\!\rangle - \Delta_2^\varepsilon, \end{aligned} \tag{2.1.30}$$

$$\text{where } \Delta_2^\varepsilon := -\langle\!\langle V_t, \mathcal{T}_\varepsilon v^\varepsilon \rangle\!\rangle - \mathbb{B}(V, \mathcal{T}_\varepsilon v^\varepsilon) + \langle\!\langle \mathbb{F}(V), \mathcal{T}_\varepsilon v^\varepsilon \rangle\!\rangle.$$

Since V is a weak solution in $\mathbb{X}$, the error Δ_2^ε would vanish, if $\mathcal{T}_\varepsilon v^\varepsilon \in \mathbb{X}$, i.e. $\mathcal{T}_\varepsilon v^\varepsilon$ would be an admissible test function. In general this is not the case, but in analogy to the $\mathcal{T}_\varepsilon$-property of recovered periodicity (1.2.13), we may assume that $V = \text{w-}\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon v^\varepsilon$ is compatible with the space $\mathbb{X}$, despite the fact that $\mathcal{T}_\varepsilon v^\varepsilon \notin \mathbb{X}$. Thus, we have $\lim_{\varepsilon\to 0} \Delta_2^\varepsilon = 0$.

Inserting (2.1.29) and (2.1.30) into (2.1.28), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2 &= -\mathbb{B}_\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon, W^\varepsilon) + \mathbb{B}(V, W^\varepsilon) + \langle\!\langle \mathcal{T}_\varepsilon F^\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon), W^\varepsilon \rangle\!\rangle - \langle\!\langle \mathbb{F}(V), W^\varepsilon \rangle\!\rangle \\ &\qquad + \Delta_1^\varepsilon + \Delta_2^\varepsilon \\ &= -\mathbb{B}_\varepsilon(W^\varepsilon, W^\varepsilon) + \langle\!\langle \mathcal{T}_\varepsilon F^\varepsilon(\mathcal{T}_\varepsilon F^\varepsilon) - \mathcal{T}_\varepsilon F^\varepsilon(V), W^\varepsilon \rangle\!\rangle + \Delta^\varepsilon, \end{aligned} \tag{2.1.31}$$

where $\Delta^\varepsilon := \sum_{i=1}^4 \Delta_i^\varepsilon$ collects also the *approximation errors* of the given data, viz.

$$\Delta_3^\varepsilon := \mathbb{B}(V, W^\varepsilon) - \mathbb{B}_\varepsilon(V, W^\varepsilon) \quad \text{and} \quad \Delta_4^\varepsilon := \langle\!\langle \mathcal{T}_\varepsilon F^\varepsilon(V), W^\varepsilon \rangle\!\rangle - \langle\!\langle \mathbb{F}(V), W^\varepsilon \rangle\!\rangle.$$

Exploiting the uniform ellipticity of $\mathbb{B}_\varepsilon$ and the global Lipschitz continuity of F^ε, equation (2.1.31) yields the Gronwall estimate (2.1.27). The detailed derivation of the error terms follows in Subsection 2.1.5.

It is then left to show that the error Δ^ε vanishes for $\varepsilon \to 0$. Together with the assumption $W^\varepsilon(0) \to 0$, one then obtains the desired result $W^\varepsilon(t) \to 0$ for all $t > 0$. This is the second part of the proof for Main Theorem I and it is carried out in Subsection 2.1.6.

2.1.5 Derivation of the error terms

This section is devoted to the derivation of the estimate (2.0.5.Est), which is crucial for the proofs of the three main theorems of this chapter. Following the strategy outlined in Subsection 2.1.4, we derive the Gronwall estimate (2.1.27) and precise the error terms.

Theorem 2.1.6 (Derivation of the error terms)**.** *Let the assumptions* (2.1.9.Exist$_\varepsilon$)*,* (2.1.9.Time$_\varepsilon$)*,* (2.1.12.Exist$_0$)*, and* (2.1.12.Time$_0$) *be satisfied. Further, let* $(u^\varepsilon, v^\varepsilon)$ *and* (u, V) *denote the solutions of* (2.0.1.P$_\varepsilon$) *and* (2.0.2.P$_0$)*, respectively. Then, there exists a constant* $C \geq 0$ *only depending on the Lipschitz constant* L *and* $T > 0$ *such that* (2.0.5.Est) *holds, namely*

$$\max_{0\leq t\leq T} \Big\{ \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t)\|^2_{\mathbb{H}_{\mathbb{R}^\mathrm{d}}} + \|u^\varepsilon(t) - u(t)\|^2_H \Big\}$$
$$\leq C \left\{ \| \mathcal{T}_\varepsilon v^\varepsilon_0 - V^{\mathrm{ex}}_0\|^2_{\mathbb{H}_{\mathbb{R}^\mathrm{d}}} + \|u^\varepsilon_0 - u_0\|^2_H + \int_0^T \Delta^{v^\varepsilon}(t) + \Delta^{u^\varepsilon}(t)\,\mathrm{d}t \right\}.$$

The error terms $\Delta^{w^\varepsilon} = \sum_{i=1}^5 |\Delta_i^{w^\varepsilon}|$ *with* $w^\varepsilon \in \{v^\varepsilon, u^\varepsilon\}$ *decompose as follows: the folding mismatch errors*

$$\Delta_1^{u^\varepsilon} := \int_\Omega (F_1^\varepsilon(u^\varepsilon, v^\varepsilon) - u_t^\varepsilon) \cdot (u - \mathcal{G}_\varepsilon^0(u, U))$$
$$- D_1^\varepsilon \nabla u^\varepsilon : \Big\{ \mathcal{F}_\varepsilon[\nabla u + \nabla_y U]^{\mathrm{ex}} - \nabla\, \mathcal{G}_\varepsilon^0(u, U) \Big\}\,\mathrm{d}x, \tag{2.1.32}$$

$$\Delta_1^{v^\varepsilon} := \int_\Omega (F_2^\varepsilon(u^\varepsilon, v^\varepsilon) - v_t^\varepsilon) \cdot (\mathcal{F}_\varepsilon\, V - \mathcal{G}_\varepsilon^1\, V)$$
$$- \varepsilon D_2^\varepsilon \nabla v^\varepsilon : \Big[\mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon \nabla(\mathcal{G}_\varepsilon^1\, V) \Big]\,\mathrm{d}x; \tag{2.1.33}$$

the periodicity defect errors

$$\Delta_2^{u^\varepsilon} := \int_\Omega (F_{\mathrm{eff}}(u, V) - u_t) \cdot u^\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathcal{Y}} \mathbb{D}_1^{\mathrm{ex}} [\nabla u + \nabla_y U]^{\mathrm{ex}} : \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1.34}$$

$$\Delta_2^{v^\varepsilon} := \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - V_t^{\mathrm{ex}}) \cdot \mathcal{T}_\varepsilon\, v^\varepsilon - \mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y(\mathcal{T}_\varepsilon\, v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y; \tag{2.1.35}$$

the approximations errors

$$\Delta_3^{u^\varepsilon} := \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathbb{D}_1^{\mathrm{ex}} - \mathcal{T}_\varepsilon\, D_1^\varepsilon)[\nabla u + \nabla_y U]^{\mathrm{ex}} : \{\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\mathrm{ex}}\}\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1.36}$$

$$\Delta_3^{v^\varepsilon} := \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathbb{D}_2^{\mathrm{ex}} - \mathcal{T}_\varepsilon\, D_2^\varepsilon) \nabla_y V^{\mathrm{ex}} : \nabla_y(\mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}})\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1.37}$$

$$\Delta_4^{u^\varepsilon} := \int_\Omega [F_1^\varepsilon(u, \mathcal{F}_\varepsilon\, V^{\mathrm{ex}}) - F_{\mathrm{eff}}(u, V)] \cdot (u^\varepsilon - u)\,\mathrm{d}x, \tag{2.1.38}$$

$$\Delta_4^{v^\varepsilon} := \int_{\mathbb{R}^d\times\mathcal{Y}} [\mathcal{T}_\varepsilon\, F_2^\varepsilon(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}})] \cdot (\mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}})\,\mathrm{d}x\,\mathrm{d}y; \tag{2.1.39}$$

and the unfolding errors due to coupling

$$\Delta_5^{u^\varepsilon} := L\|V^{\mathrm{ex}} - \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V^{\mathrm{ex}}\|^2_{\mathbb{H}_{\mathbb{R}^d}} \quad \textit{and} \quad \Delta_5^{v^\varepsilon} := L\| \mathcal{T}_\varepsilon\, u - u^{\mathrm{ex}}\|^2_{\mathbb{H}_{\mathbb{R}^d}}. \tag{2.1.40}$$

Remark 2.1.7. *We point out that for the proof of Theorem 2.1.6, we do not rely on the continuity in time of ∇v^ε and v_t^ε. Indeed, we only need that $v_t^\varepsilon(t)$ and $V_t(t)$ are L^2-integrable in space for almost all $t \in [0,T]$ to apply $\mathcal{T}_\varepsilon$ to v_t^ε.*

Before entering the details of the proof, let us now sketch its main steps: We derive in two separate parts of the proof the Gronwall-type estimates

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 \le C\{\|u^\varepsilon - u\|_H^2 + \|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \Delta^{u^\varepsilon}\} \tag{2.1.41}$$

as well as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 \le C\{\|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|u^\varepsilon - u\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \Delta^{v^\varepsilon}\}, \tag{2.1.42}$$

where both inequalities hold true for every time point $t \in [0,T]$. Adding (2.1.41) and (2.1.42) and applying Gronwall's lemma gives (2.0.5.Est). Below, we explain in more detail the derivation of (2.1.42); and (2.1.41) follows analogously. Therefore, let us consider

$$\begin{aligned} v_t^\varepsilon \quad &= \operatorname{div}(\varepsilon^2 D_2^\varepsilon \nabla v^\varepsilon) + F_2^\varepsilon(u^\varepsilon, v^\varepsilon) && \text{in } [0,T]\times\Omega \\ v^\varepsilon(0) &= v_0^\varepsilon && \text{in } \Omega, \end{aligned} \tag{2.1.43}$$

denoting the v^ε-equations in $(2.1.7.\mathrm{P}_\varepsilon)_2$. The limiting equations in $(2.1.11.\mathrm{P}_0)_2$ read

$$\begin{aligned} V_t \quad &= \operatorname{div}_y(\mathbb{D}_2\nabla_y V) + \mathbb{F}_2(u,V) && \text{in } [0,T]\times\Omega\times\mathcal{Y} \\ V(0) &= V_0 && \text{in } \Omega\times\mathcal{Y}. \end{aligned} \tag{2.1.44}$$

The following Steps 1–3 are applied to (2.1.43)–(2.1.44) and, afterward, they are repeated for the u^ε- respective u-equations in order to derive the Gronwall-type estimate (2.1.41).

Step 1: Reformulation of (2.1.43) *and specification of the folding mismatch $\Delta_1^{v^\varepsilon}$.* The underlying domains of the ε-problem (2.1.43) and the effective one (2.1.44) are Ω and $\Omega\times\mathcal{Y}$. To subtract their weak formulations, as in (2.1.28)–(2.1.30), we unfold the ε-problem to the common domain of integration $\mathbb{R}^d\times\mathcal{Y}$ by using the folding and unfolding operators from Subsection 1.2.2 and 1.2.4. Inserting a suitable test function, we arrive at the definition of the folding mismatch $\Delta_1^{v^\varepsilon}$ as specified in (2.1.29).

Step 2: Specification of the periodicity defect error $\Delta_2^{v^\varepsilon}$. We derive equation (2.1.30) and the exact form of the error term $\Delta_2^{v^\varepsilon}$ induced by the periodicity defect of $\mathcal{T}_\varepsilon v^\varepsilon$. The error terms $\Delta_1^{v^\varepsilon}$ and $\Delta_2^{v^\varepsilon}$ look in principle as in Subsection 2.1.4, but are a little more involved owing to the precise definition of the folding and unfolding operators.

Step 3: Preparation of the Gronwall estimate and the approximation errors $\Delta_3^{v^\varepsilon}$–$\Delta_5^{v^\varepsilon}$. As in (2.1.27) and (2.1.31), we subtract the reformulated weak formulations of (2.1.43) and (2.1.44), derived in Step 2–3, and we precise the error terms $\Delta_3^{v^\varepsilon}$–$\Delta_5^{v^\varepsilon}$, which contain the approximation errors $D^\varepsilon \rightsquigarrow \mathbb{D}$ and $F^\varepsilon \rightsquigarrow \mathbb{F}$.

Step 4: Estimation via Gronwall's lemma.

Proof of Theorem 2.1.6. We begin with deriving (2.1.42). For all $t \in [0,T]$, we set

$$W^\varepsilon(t) := \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t),$$

where $W^\varepsilon \in C([0,T];\widetilde{\mathbb{X}}) \cap C^1([0,T];\mathbb{H}_{\mathbb{R}^d})$ with $\widetilde{\mathbb{X}} = \mathrm{L}^2(\mathbb{R}^d;\mathrm{H}^1(Y))$ according to Theorem 1.2.2(a) and improved time-regularity.

Step 1(a): Reformulation of (2.1.43) *and specification of the folding mismatch* $\Delta_1^{v^\varepsilon}$. Let $t \in [0,T]$ be arbitrary but fixed and let all upcoming equations hold for all $t \in [0,T]$, if not stated otherwise. The weak formulation of (2.1.43) reads

$$\int_\Omega v_t^\varepsilon \cdot \varphi \,\mathrm{d}x = \int_\Omega -D_2^\varepsilon \varepsilon \nabla v^\varepsilon : \varepsilon \nabla \varphi + F_2^\varepsilon(u^\varepsilon, v^\varepsilon)\cdot \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in X_\varepsilon. \tag{2.1.45}$$

Let $V \in W_{\mathrm{imp}}(0,T;\mathbb{X})$ be the unique weak solution of (2.1.44) and we choose the test function $\varphi^\varepsilon = v^\varepsilon - \mathcal{G}_\varepsilon^1 V \in X_\varepsilon$, see Definition 1.2.7 for $\mathcal{G}_\varepsilon^1$. Using the identity $\mathcal{F}_\varepsilon \mathcal{T}_\varepsilon = \mathrm{id}|_H$ (Proposition 1.2.1) and adding $\pm \mathcal{F}_\varepsilon V$ resp. $\pm \mathcal{F}_\varepsilon(\nabla_y V)$, we obtain

$$\begin{aligned}\int_\Omega v_t^\varepsilon \cdot \mathcal{F}_\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon - V)\,\mathrm{d}x = \int_\Omega & -D_2^\varepsilon \varepsilon\nabla v^\varepsilon : \mathcal{F}_\varepsilon\left[\mathcal{T}_\varepsilon(\varepsilon \nabla v^\varepsilon) - \nabla_y V\right] \\ & + F_2^\varepsilon(u^\varepsilon, v^\varepsilon)\cdot \mathcal{F}_\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon - V)\,\mathrm{d}x + \Delta_1^{v^\varepsilon}\end{aligned} \tag{2.1.46}$$

with the folding mismatch error (2.1.33), viz.

$$\Delta_1^{v^\varepsilon} := \int_\Omega (F_2^\varepsilon(u^\varepsilon,v^\varepsilon) - v_t^\varepsilon)\cdot(\mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon^1 V) - \varepsilon D_2^\varepsilon \nabla v^\varepsilon : \Big[\mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon\nabla(\mathcal{G}_\varepsilon^1 V)\Big]\,\mathrm{d}x.$$

Since $\mathcal{T}_\varepsilon$ is a linear and bounded operator, it commutes with differentiation, i.e. $\mathcal{T}_\varepsilon(v_t^\varepsilon) = (\mathcal{T}_\varepsilon v^\varepsilon)_t$. Exploiting the duality $\mathcal{F}_\varepsilon{}' = \mathcal{T}_\varepsilon$, as well as $\mathcal{T}_\varepsilon[D_2^\varepsilon \varepsilon \nabla v^\varepsilon] = \mathcal{T}_\varepsilon D_2^\varepsilon \mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon)$, $\mathcal{T}_\varepsilon[F_2^\varepsilon(u^\varepsilon,v^\varepsilon)] = \mathcal{T}_\varepsilon F_2^\varepsilon(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon v^\varepsilon)$ and $\mathcal{T}_\varepsilon(\varepsilon \nabla v^\varepsilon) = \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon)$ (cf. Theorem 1.2.2 and (1.2.6)), we arrive at

$$\begin{aligned}&\int\limits_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon v^\varepsilon)_t \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y = \\ &\qquad \int\limits_{\mathbb{R}^d\times\mathcal{Y}} -\mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon) : \nabla_y W^\varepsilon + \mathcal{T}_\varepsilon F_2^\varepsilon(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon v^\varepsilon)\cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y + \Delta_1^{v^\varepsilon}.\end{aligned} \tag{2.1.47}$$

Hence, the reformulation of (2.1.43) is completed, and (2.1.29) is established with

$$\mathbb{B}_\varepsilon(V,W) = \int_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y V : \nabla_y W \,\mathrm{d}x\,\mathrm{d}y.$$

Step 2(a): Specification of the periodicity defect error $\Delta_2^{v^\varepsilon}$*.* Next we consider the weak formulation of the effective equations (2.1.44)

$$\int_{\Omega\times\mathcal{Y}} V_t \cdot \Phi \,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times\mathcal{Y}} -\mathbb{D}_2 \nabla_y V : \nabla_y \Phi + \mathbb{F}_2(u,V)\cdot\Phi\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \Phi \in \mathbb{X}. \tag{2.1.48}$$

We aim to derive (2.1.30), but we observe a discrepancy in the domains of integration in (2.1.47) and (2.1.48). Therefore we reformulate (2.1.48) by extending all the functions by 0 outside of Ω, i.e.

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathcal{Y}} V_t^{\mathrm{ex}} \cdot \Phi\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d\times\mathcal{Y}} -\mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y\Phi + \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}})\cdot\Phi\,\mathrm{d}x\,\mathrm{d}y \\ \text{for all } \Phi \in \mathrm{L}^2(\mathbb{R}^d;\mathrm{H}^1(\mathcal{Y})).\end{aligned} \tag{2.1.49}$$

Here, the regularity $u^{\mathrm{ex}} \in \mathrm{L}^2(\mathbb{R})$ is sufficient, since $\mathbb{F}_2^{\mathrm{ex}}$ maps L^2-functions to L^2-functions. Although $\Phi = \mathcal{T}_\varepsilon v^\varepsilon$ is not admissible in (2.1.49) because of the periodicity defect, each integral expression in (2.1.49), considered on its own, is well-defined for $\Phi = \mathcal{T}_\varepsilon v^\varepsilon$. Because of this, we test (2.1.49) with $\Phi = V^{\mathrm{ex}}$ only and then add and subtract the missing terms $-V_t^{\mathrm{ex}} \cdot \mathcal{T}_\varepsilon v^\varepsilon + \mathbb{D}_2 \nabla_y V^{\mathrm{ex}} : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \cdot \mathcal{T}_\varepsilon v^\varepsilon$ at the cost of creating the periodicity defect error $\Delta_2^{v^\varepsilon}$:

$$\begin{aligned}
&-\int_{\mathbb{R}^d \times \mathcal{Y}} V_t^{\mathrm{ex}} \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d \times \mathcal{Y}} V_t^{\mathrm{ex}} \cdot V^{\mathrm{ex}} \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^d \times \mathcal{Y}} V_t^{\mathrm{ex}} \cdot \mathcal{T}_\varepsilon v^\varepsilon \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d \times \mathcal{Y}} -\mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y V^{\mathrm{ex}} + \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \cdot V^{\mathrm{ex}} \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^d \times \mathcal{Y}} V_t^{\mathrm{ex}} \cdot \mathcal{T}_\varepsilon v^\varepsilon \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d \times \mathcal{Y}} \mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y W^\varepsilon - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y + \Delta_2^{v^\varepsilon}, \qquad (2.1.50)
\end{aligned}$$

where $\Delta_2^{v^\varepsilon}$ is given by (2.1.35), viz.

$$\Delta_2^{v^\varepsilon} := \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - V_t^{\mathrm{ex}}) \cdot \mathcal{T}_\varepsilon v^\varepsilon - \mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon) \,\mathrm{d}x\,\mathrm{d}y.$$

Thus, (2.1.30) is established.

Step 3(a): Preparation of the Gronwall estimate and the approximation errors $\Delta_3^{v^\varepsilon}$–$\Delta_5^{v^\varepsilon}$. In order to apply Gronwall's lemma in Step 4, we now prepare the estimate (2.1.27). We begin by adding (2.1.47) and (2.1.50), as suggested in (2.1.28) and (2.1.31),

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2_{\mathbb{H}_{\mathbb{R}^d}} &= \int_{\mathbb{R}^d \times \mathcal{Y}} W_t^\varepsilon \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathcal{T}_\varepsilon v^\varepsilon)_t \cdot W^\varepsilon - V_t^{\mathrm{ex}} \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d \times \mathcal{Y}} -\mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon) : \nabla_y W^\varepsilon + \mathcal{T}_\varepsilon F^\varepsilon(\mathcal{T}_\varepsilon v^\varepsilon) \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y + \Delta_1^{v^\varepsilon} \\
&\quad + \int_{\mathbb{R}^d \times \mathcal{Y}} \mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y W^\varepsilon - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y + \Delta_2^{v^\varepsilon}. \qquad (2.1.51)
\end{aligned}$$

Rewriting the gradient terms via

$$\begin{aligned}
&-\mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon) : \nabla_y W^\varepsilon + \mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y W^\varepsilon \\
&= -\mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y W^\varepsilon : \nabla_y W^\varepsilon + (\mathbb{D}_2^{\mathrm{ex}} - \mathcal{T}_\varepsilon D_2^\varepsilon) \nabla_y V^{\mathrm{ex}} : \nabla_y W^\varepsilon, \qquad (2.1.52)
\end{aligned}$$

equation (2.1.51) takes the form

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2_{\mathbb{H}_{\mathbb{R}^d}} \\
&= \int_{\mathbb{R}^d \times \mathcal{Y}} -\mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y W^\varepsilon : \nabla_y W^\varepsilon + \left[\mathcal{T}_\varepsilon F_2^\varepsilon(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon v^\varepsilon) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}})\right] \cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \Delta_1^{v^\varepsilon} + \Delta_2^{v^\varepsilon} + \Delta_3^{v^\varepsilon}, \qquad (2.1.53) \\
&\quad \text{where} \quad \Delta_3^{v^\varepsilon} := \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathbb{D}_2^{\mathrm{ex}} - \mathcal{T}_\varepsilon D_2^\varepsilon) \nabla_y V^{\mathrm{ex}} : \nabla_y W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Analogously we rearrange the reaction terms in (2.1.53) via

$$\begin{aligned}
&\mathcal{T}_\varepsilon F_2^\varepsilon(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon v^\varepsilon) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \\
&= [\mathcal{T}_\varepsilon F_2^\varepsilon(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon v^\varepsilon) - \mathcal{T}_\varepsilon F_2^\varepsilon(u^{\mathrm{ex}}, V^{\mathrm{ex}})] + [\mathcal{T}_\varepsilon F_2^\varepsilon(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}})] \qquad (2.1.54)
\end{aligned}$$

and we define the approximation error $\Delta_4^{v^\varepsilon}$ via

$$\Delta_4^{v^\varepsilon} := \int_{\mathbb{R}^d\times\mathcal{Y}} [\mathcal{T}_\varepsilon\, F_2^\varepsilon(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}})]\cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad \Delta_*^{v^\varepsilon} := \sum_{i=1}^4 \Delta_i^{v^\varepsilon}.$$

Note, $\Delta^\varepsilon(t)$ depends on the time variable t via the given data and solutions. Applying the reformulation (2.1.54), the ellipticity of D_2^ε, and the global Lipschitz continuity of F_2^ε to (2.1.53) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2_{\mathbb{H}_{\mathbb{R}^d}}\\
&= \int_{\mathbb{R}^d\times\mathcal{Y}} -\mathcal{T}_\varepsilon\, D_2^\varepsilon \nabla_y W^\varepsilon{:}\nabla_y W^\varepsilon + [\mathcal{T}_\varepsilon\, F_2^\varepsilon(\mathcal{T}_\varepsilon\, u^\varepsilon, \mathcal{T}_\varepsilon\, v^\varepsilon) - \mathcal{T}_\varepsilon\, F_2^\varepsilon(u^{\mathrm{ex}}, V^{\mathrm{ex}})]\cdot W^\varepsilon \,\mathrm{d}x\,\mathrm{d}y + \Delta_*^{v^\varepsilon}\\
&\le -\alpha\|\nabla_y W^\varepsilon\|^2_{\mathbb{H}_{\mathbb{R}^d}} + L\left\{\|W^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}} + \|\,\mathcal{T}_\varepsilon\, u^\varepsilon - u^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}\right\}\|W^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}} + \Delta_*^{v^\varepsilon}.
\end{aligned} \tag{2.1.55}$$

It holds $\mathcal{T}_\varepsilon\, D_2^\varepsilon(t,x,y)\xi : \xi \ge \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^{m_2\times d}$ and all $(t,x,y) \in [0,T]\times[\Omega\times\mathcal{Y}]^\varepsilon$, where $\mathrm{supp}(\mathcal{T}_\varepsilon\, v^\varepsilon) \subset \overline{[\Omega\times\mathcal{Y}]^\varepsilon}$. Inserting $\pm\,\mathcal{T}_\varepsilon\, u$ and applying once more the triangle and Young's inequality in (2.1.55) as well as setting

$$\Delta_5^{v^\varepsilon} := L\|\,\mathcal{T}_\varepsilon\, u - u^{\mathrm{ex}}\|^2_{\mathbb{H}_{\mathbb{R}^d}} \quad \text{and} \quad \Delta^{v^\varepsilon} := \sum_{i=1}^5 \left|\Delta_i^{v^\varepsilon}\right|,$$

we arrive at (2.1.27), viz.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|^2_{\mathbb{H}_{\mathbb{R}^d}} \le \frac{3}{2}L\left\{\|W^\varepsilon\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon - u\|^2_H\right\} + \Delta^{v^\varepsilon}. \tag{2.1.56}$$

Hence, the estimate (2.1.42) is established. In what follows, we repeat Step 1–3 for the u^ε- respective u-equations and derive (2.1.41).

Step 1(b). We test the u^ε-equations in $(2.1.7)_1$

$$\int_\Omega u_t^\varepsilon\cdot\psi\,\mathrm{d}x = \int_\Omega -D_1^\varepsilon\nabla u^\varepsilon : \nabla\psi + F_1^\varepsilon(u^\varepsilon, v^\varepsilon)\cdot\psi\,\mathrm{d}x \quad \text{for all } \psi\in X \tag{2.1.57}$$

with $\psi = u^\varepsilon - \mathcal{G}_\varepsilon^0(u,U)$, where $u \in W_{\mathrm{imp}}(0,T;X)$ solves the effective equations in $(2.1.11.\mathrm{P}_0)_1$ and $U \in C([0,T];\mathbb{X}_0)$ is the unique corrector according to (2.1.20) and (2.1.23). Moreover, inserting the terms $\pm u$ resp. $\pm\,\mathcal{F}_\varepsilon[\nabla u + \nabla_y U]^{\mathrm{ex}}$ and rearranging gives

$$\begin{aligned}
\int_\Omega u_t^\varepsilon\cdot(u^\varepsilon - u)\,\mathrm{d}x = \int_\Omega &-D_1^\varepsilon\nabla u^\varepsilon : \{\nabla u^\varepsilon - \mathcal{F}_\varepsilon[\nabla u + \nabla_y U]^{\mathrm{ex}}\}\\
&+ F_1^\varepsilon(u^\varepsilon, v^\varepsilon)\cdot(u^\varepsilon - u)\,\mathrm{d}x + \Delta_1^{u^\varepsilon},
\end{aligned}$$

$$\begin{aligned}
\text{where } \Delta_1^{u^\varepsilon} := \int_\Omega &(F_1^\varepsilon(u^\varepsilon, v^\varepsilon) - u_t^\varepsilon)\cdot(u - \mathcal{G}_\varepsilon^0(u,U))\\
&- D_1^\varepsilon\nabla u^\varepsilon : \{\mathcal{F}_\varepsilon[\nabla u + \nabla_y U]^{\mathrm{ex}} - \nabla\,\mathcal{G}_\varepsilon^0(u,U)\}\,\mathrm{d}x.
\end{aligned}$$

Using the duality of $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$ as well as $\mathcal{F}_\varepsilon\,\mathcal{T}_\varepsilon = \mathrm{id}|_H$, we have

$$\begin{aligned}
\int_\Omega u_t^\varepsilon\cdot(u^\varepsilon - u)\,\mathrm{d}x = &\int_{\mathbb{R}^d\times\mathcal{Y}} -\mathcal{T}_\varepsilon\, D_1^\varepsilon\,\mathcal{T}_\varepsilon(\nabla u^\varepsilon) : \{\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\mathrm{ex}}\}\,\mathrm{d}x\,\mathrm{d}y\\
&+ \int_\Omega F_1^\varepsilon(u^\varepsilon, v^\varepsilon)\cdot(u^\varepsilon - u)\,\mathrm{d}x + \Delta_1^{u^\varepsilon}.
\end{aligned} \tag{2.1.58}$$

Step 2(b). We reformulate the u-equations in $(2.1.11.\mathrm{P}_0)_1$ using the two-scale representation for D_{eff}, see (2.1.19),

$$\int_\Omega u_t \cdot \psi \,\mathrm{d}x = \int_{\Omega\times\mathcal{Y}} -\mathbb{D}_1[\nabla u + \nabla_y U] : [\nabla\psi + \nabla_y \Psi] \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega F_{\mathrm{eff}}(u,V)\cdot\psi\,\mathrm{d}x$$
$$\text{for all } (\psi,\Psi) \in X \times \mathbb{X}_0 \quad (2.1.59)$$

by testing with the solution itself $(\psi,\Psi) = (u,U)$. Introducing the terms $\pm u^\varepsilon$ and $\pm\,\mathcal{T}_\varepsilon(\nabla u^\varepsilon)$, extending the two-scale space $\Omega\times\mathcal{Y}$ to $\mathbb{R}^d\times\mathcal{Y}$, and rearranging gives

$$\begin{aligned}\int_\Omega u_t\cdot(u-u^\varepsilon)\,\mathrm{d}x &= \int_{\mathbb{R}^d\times\mathcal{Y}} -\mathbb{D}_1^{\mathrm{ex}}[\nabla u+\nabla_y U]^{\mathrm{ex}} : \{[\nabla u+\nabla_y U]^{\mathrm{ex}} - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \int_\Omega F_{\mathrm{eff}}(u,V)\cdot(u-u^\varepsilon)\,\mathrm{d}x + \Delta_2^{u^\varepsilon}, \qquad (2.1.60)\end{aligned}$$
$$\text{where } \Delta_2^{u^\varepsilon} := \int_\Omega (F_{\mathrm{eff}}(u,V)-u_t)\cdot u^\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathcal{Y}} \mathbb{D}_1^{\mathrm{ex}}[\nabla u+\nabla_y U]^{\mathrm{ex}} : \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y.$$

Recall that the two-scale function $\nabla u + \nabla_y U$ is square-integrable w.r.t. $x\in\Omega$ and hence can be extended with 0 on $\mathbb{R}^d\backslash\Omega$ without loss of regularity.

Step 3(b). We add (2.1.58) and (2.1.60) and we rearrange as in (2.1.52) and (2.1.54) so that we arrive at

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon-u\|_H^2 &= \int_\Omega (u^\varepsilon-u)_t\cdot(u^\varepsilon-u)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d\times\mathcal{Y}} -\mathcal{T}_\varepsilon D_1^\varepsilon\{\mathcal{T}_\varepsilon(\nabla u^\varepsilon)-[\nabla u+\nabla_y U]^{\mathrm{ex}}\} : \{\mathcal{T}_\varepsilon(\nabla u^\varepsilon)-[\nabla u+\nabla_y U]^{\mathrm{ex}}\}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \int_\Omega [F_1^\varepsilon(u^\varepsilon,v^\varepsilon) - F_1^\varepsilon(u,\mathcal{F}_\varepsilon V^{\mathrm{ex}})]\cdot(u^\varepsilon-u)\,\mathrm{d}x + \Delta_*^{u^\varepsilon}, \qquad (2.1.61)\end{aligned}$$

where $\Delta_*^{u^\varepsilon} := \sum_{i=1}^4 \Delta_i^{u^\varepsilon}$ with

$$\begin{aligned}\Delta_3^{u^\varepsilon} &:= \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathbb{D}_1^{\mathrm{ex}} - \mathcal{T}_\varepsilon D_1^\varepsilon)[\nabla u+\nabla_y U]^{\mathrm{ex}} : \{\mathcal{T}_\varepsilon(\nabla u^\varepsilon)-[\nabla u+\nabla_y U]^{\mathrm{ex}}\}\,\mathrm{d}x\,\mathrm{d}y,\\ \Delta_4^{u^\varepsilon} &:= \int_\Omega [F_1^\varepsilon(u,\mathcal{F}_\varepsilon V^{\mathrm{ex}}) - F_{\mathrm{eff}}(u,V)]\cdot(u^\varepsilon-u)\,\mathrm{d}x.\end{aligned}$$

Exploiting the ellipticity of $\mathcal{T}_\varepsilon D_1^\varepsilon$ and the Lipschitz continuity of F_1^ε in (2.1.61), we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon-u\|_H^2\\ &\le -\alpha\|\mathcal{T}_\varepsilon(\nabla u^\varepsilon)-[\nabla u+\nabla_y U]^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2\\ &\quad + L\{\|u^\varepsilon-u\|_H + \|v^\varepsilon-\mathcal{F}_\varepsilon V^{\mathrm{ex}}\|_H\}\|u^\varepsilon-u\|_H + \Delta_*^{u^\varepsilon} \qquad (2.1.62)\\ &\le \tfrac{3}{2}L\{\|u^\varepsilon-u\|_H^2 + \|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2\} + \Delta^{u^\varepsilon},\end{aligned}$$
$$\text{where} \quad \Delta_5^{u^\varepsilon} := L\|V^{\mathrm{ex}} - \mathcal{T}_\varepsilon\mathcal{F}_\varepsilon V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 \quad \text{and} \quad \Delta^{u^\varepsilon} = \sum_{i=1}^5 \left|\Delta_i^{u^\varepsilon}\right|.$$

Here, we used for the second estimate in (2.1.62) that $\|v^\varepsilon - \mathcal{F}_\varepsilon V^{\mathrm{ex}}\|_H = \|\mathcal{T}_\varepsilon v^\varepsilon - \mathcal{T}_\varepsilon\mathcal{F}_\varepsilon V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} \le \|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} + \|V^{\mathrm{ex}} - \mathcal{T}_\varepsilon\mathcal{F}_\varepsilon V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}$. Hence, the Gronwall-type estimate (2.1.41) is established.

Step 4: Estimation via Gronwall's lemma. Adding (2.1.41) and (2.1.42) and integrating over $[0,t]$ for $0 < t \leq T$ gives

$$
\begin{aligned}
&\| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\text{ex}}(t)\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(t) - u(t)\|^2_H \\
&\leq C \Big\{ \| \mathcal{T}_\varepsilon v^\varepsilon(0) - V^{\text{ex}}(0)\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(0) - u(0)\|^2_H \\
&+ \int_0^t \| \mathcal{T}_\varepsilon v^\varepsilon(s) - V^{\text{ex}}(s)\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(s) - u(s)\|^2_H + \Delta^{v^\varepsilon}(s) + \Delta^{u^\varepsilon}(s)\,\mathrm{d}s \Big\},
\end{aligned}
$$

where the constant $C \geq 0$ is a multiple of L. We point out that solutions are even without improved time-regularity continuous in time with values in H respective $\mathbb{H}$ and the error terms are by construction nonnegative and integrable on $[0,T]$. Hence, the application of Gronwall's lemma yields for all $t \in [0,T]$

$$
\begin{aligned}
&\| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\text{ex}}(t)\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(t) - u(t)\|^2_H \\
&\leq e^{CT} \Big\{ \| \mathcal{T}_\varepsilon v^\varepsilon(0) - V^{\text{ex}}(0)\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(0) - u(0)\|^2_H + \int_0^t \Delta^{v^\varepsilon}(s) + \Delta^{u^\varepsilon}(s)\,\mathrm{d}s \Big\}. \qquad (2.1.63)
\end{aligned}
$$

Taking the maximum over all $t \in [0,T]$ finishes the proof of (2.0.5.Est). □

2.1.6 Control of the error terms and proof of Main Theorem I

In this subsection, we carry out the proof of Theorem 2.1.1 based on estimate (2.0.5.Est), which is derived in Theorem 2.1.6, cf. (2.1.63) above. Let the assumptions of Theorem 2.1.1 hold and let $(u^\varepsilon, v^\varepsilon)$ and (u, V) denote the solutions of (2.1.7.P$_\varepsilon$) and (2.1.11.P$_0$), respectively. We prove in the following the convergences (2.1.15a)–(2.1.15d) of the sequence $(u^\varepsilon, v^\varepsilon)_\varepsilon$ to (u, V), repeated below:

$$
\begin{aligned}
&\text{uniformly for all } t \in [0,T]:\ v^\varepsilon(t) \xrightarrow{2\text{s}} V(t) \text{ in } \mathbb{H}, \\
&\text{i.e. } \max\nolimits_{0\leq t\leq T} \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\text{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} \to 0, && \text{(a)}\\
&\text{pointwise for all } t \in [0,T]:\ \varepsilon\nabla v^\varepsilon(t) \xrightarrow{2\text{s}} \nabla_y V(t) \text{ in } \mathbb{H} \\
&\text{and } \varepsilon\nabla v^\varepsilon \xrightarrow{2\text{s}} \nabla_y V \text{ in } \mathrm{L}^2(0,T;\mathbb{H}), && \text{(b)}\\
&v^\varepsilon_t \xrightharpoonup{2\text{w}} V_t \text{ in } \mathrm{L}^2(0,T;\mathbb{H}), && \text{(c)}\\
&u^\varepsilon \rightharpoonup u \text{ in } \mathrm{L}^2(0,T;X) \text{ and } u^\varepsilon_t \rightharpoonup u_t \text{ in } \mathrm{L}^2(0,T;X^*), \\
&\text{moreover, there exists a function } U \in \mathrm{L}^2(0,T;\mathbb{X}_0) \text{ such that} && \text{(d)}\\
&\text{pointwise for all } t \in [0,T]:\ \nabla u^\varepsilon(t) \xrightarrow{2\text{s}} \nabla u(t) + \nabla_y U(t) \text{ in } \mathbb{H}.
\end{aligned}
$$

The proof of Theorem 2.1.1 is structured as follows.

Step 1: Extraction of weakly convergent subsequences. The a priori bounds (2.1.10) allow us to extract a weakly converging subsequence of $(u^\varepsilon, v^\varepsilon)_\varepsilon$. In particular, we obtain for $(v^\varepsilon)_\varepsilon$ a limit function $\widetilde{V}$. At this point we exploit the improved time-regularity of the sequence $(v^\varepsilon)_\varepsilon$; we apply Arzèla–Ascoli's theorem in order to obtain one subsequence such that for all times $v^\varepsilon \xrightharpoonup{2\text{w}} \widetilde{V}$. By improving the convergence from weak to strong in the subsequent steps, we are able to show that $\widetilde{V}$ equals the unique solution V of (2.1.11.P$_0$)$_2$ and to conclude the convergence of the whole sequence.

Step 2 + 3: Controlling the error terms Δ^{w^ε} with $w^\varepsilon \in \{u^\varepsilon, v^\varepsilon\}$. To show that the right-hand side in (2.0.5.Est) vanishes as $\varepsilon \to 0$, we provide an ε-independent and integrable majorant for each $\Delta_i^{w^\varepsilon} : [0,T] \to [0,\infty)$ and we show further $\lim_{\varepsilon\to 0} \Delta_i^{w^\varepsilon}(t) = 0$ pointwise for all $t \in [0,T]$ as $\varepsilon \to 0$. In particular, we obtain

$$\lim_{\varepsilon\to 0} \int_0^T \Delta^{v^\varepsilon}(t) + \Delta^{u^\varepsilon}(t)\,\mathrm{d}t = 0 \tag{2.1.65}$$

and the strong two-scale convergence (2.1.15a) follows using the convergence of the initial values, see (2.1.14.Conv).

Step 4: Derivation of the remaining convergences (2.1.15b)–(2.1.15d).

Proof of Theorem 2.1.1. *Step 1: Extraction of weakly convergent subsequences.* Let $(u^\varepsilon, v^\varepsilon)$ be the weak solution of (2.1.7.P_ε) satisfying the uniform bounds (2.1.10). Applying Banach's selection principle yields the existence of a subsequence ε' of ε and a limit function $u \in W_{\mathrm{imp}}0,T;X)$ such that $u^{\varepsilon'} \rightharpoonup u$ in $\mathrm{H}^1(0,T;X)$ and $u_t^{\varepsilon'} \rightharpoonup u_t$ in $\mathrm{H}^1(0,T;X^*)$ hold true. Moreover, Theorem 1.2.5(c) yields $U \in \mathrm{L}^2(0,T;\mathbb{X}_0)$ such that $\nabla u^{\varepsilon'} \xrightharpoonup{2\mathrm{w}} \nabla u + \nabla_y U$ in $\mathrm{L}^2(0,T;\mathbb{H})$.

In particular, $\|u^\varepsilon\|_{\mathrm{H}^1(0,T;X)} \le C_b$ and $\mathrm{H}^1(0,T;X)$ is continuously embedded into the Hölder space $C^{1/2}([0,T];X)$. Using $X \subset H$ compact, the Arzèla–Ascoli theorem yields

$$\forall\, t \in [0,T]: \quad u^{\varepsilon'}(t) \to u(t) \quad \text{in } H. \tag{2.1.66}$$

As mentioned before, the strong convergence of u^ε is immediate. Nevertheless, we do all calculations that we do for v^ε for u^ε as well in order to show $\nabla u^\varepsilon \xrightarrow{2\mathrm{s}} \nabla u + \nabla_y U$.

The uniform bound (2.1.10) implies $\|v^\varepsilon\|_{\mathrm{H}^1(0,T;H)} + \|\varepsilon \nabla v^\varepsilon\|_{\mathrm{H}^1(0,T;H)} \le C_b$. On the one hand, Theorem 1.2.5(b) yields for all $t \in [0,T]$ the existence of a subsequence $\varepsilon'(t)$ of ε and a limit $\widetilde{V}(t) \in \mathbb{X}$ such that $v^{\varepsilon'(t)}(t) \xrightharpoonup{2\mathrm{w}} \widetilde{V}(t)$ and $\varepsilon' \nabla v^{\varepsilon'(t)}(t) \xrightharpoonup{2\mathrm{w}} \nabla_y \widetilde{V}(t)$ in $\mathbb{H}$. On the other hand, we have $\|\mathcal{T}_\varepsilon v^\varepsilon\|_{\mathrm{H}^1(0,T;\widetilde{\mathbb{X}})} \le C_b$, where we use the abbreviation $\widetilde{\mathbb{X}} := \mathrm{L}^2(\mathbb{R}^d;\mathrm{H}^1(Y))$ and set for all $t \in [0,T]$

$$z^\varepsilon(t) := \big(\mathcal{T}_\varepsilon v^\varepsilon(t), \Phi(t)\big)_{\widetilde{\mathbb{X}}} \quad \text{for arbitrary } \Phi \in C^\infty([0,T];\widetilde{\mathbb{X}}).$$

We observe that $z^\varepsilon : [0,T] \to \mathbb{R}$ is uniformly bounded in $C^{1/2}([0,T];\mathbb{R})$ w.r.t. all $\varepsilon > 0$. Hence, we can apply the Arzelà–Ascoli theorem so that we find a subsequence ε' of ε and a limit $z \in C([0,T];\mathbb{R})$ such that $\forall\, t \in [0,T] : \ z^{\varepsilon'}(t) \to z(t)$. Overall, we obtain $\widetilde{V} \in C([0,T];\mathbb{X})$ such that

$$\forall\, t \in [0,T]: \quad v^{\varepsilon'}(t) \xrightharpoonup{2\mathrm{w}} \widetilde{V}(t) \ \text{ and } \ \varepsilon' \nabla v^{\varepsilon'}(t) \xrightharpoonup{2\mathrm{w}} \nabla_y \widetilde{V}(t) \quad \text{in } \mathbb{H}. \tag{2.1.67}$$

For the subsequent steps we resort to working with the above extracted subsequence, labeling it by ε again for notational simplicity.

Step 2: Controlling Δ^{v^ε}. For the folding mismatch error $\Delta_1^{v^\varepsilon}$, we apply Hölder's inequality, the growth conditions (1.1.5) and (1.1.8) on D_2^ε and F_2^ε, respectively, and the uniform boundedness (2.1.10) of the solutions $(u^\varepsilon, v^\varepsilon)$ so that we arrive at

$$
\begin{aligned}
|\Delta_1^{v^\varepsilon}(t)| = \Big| \int_\Omega & [F_2^\varepsilon(t, u^\varepsilon(t), v^\varepsilon(t)) - v_t^\varepsilon(t)] \cdot (\mathcal{F}_\varepsilon V(t) - \mathcal{G}_\varepsilon^1 V(t)) \\
& - D_2^\varepsilon(t)\varepsilon\nabla v^\varepsilon(t) : (\mathcal{F}_\varepsilon[\nabla_y V(t)] - \varepsilon\nabla[\mathcal{G}_\varepsilon^1 V(t)])\,\mathrm{d}x \Big| \\
\le C(C_b, C_1, \beta) & \Big(\| \mathcal{F}_\varepsilon V(t) - \mathcal{G}_\varepsilon^1 V(t)\|_H + \| \mathcal{F}_\varepsilon[\nabla_y V(t)] - \varepsilon\nabla[\mathcal{G}_\varepsilon^1 V(t)]\|_H \Big).
\end{aligned}
$$

By the a priori boundedness of the limit solutions (2.1.13), we find the uniform L^∞–majorant $|\Delta_1^{v^\varepsilon}(t)| \le C(C_b, C_1, \beta)$. Moreover, Proposition 1.2.9 (comparison of $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^1$) guarantees the pointwise convergence $\Delta_1^{v^\varepsilon}(t) \xrightarrow{\varepsilon\to 0} 0$ for all t.

For the periodicity defect error $\Delta_2^{v^\varepsilon}$ the uniform bounds (2.1.10) and (2.1.13) as well as the growth conditions on the given data provide an L^∞–majorant, whereas the pointwise convergence follows form the $\mathcal{T}_\varepsilon$-property of recovered periodicity (1.2.13), since $\widetilde{V}(t) \in \mathbb{X}$ obtained in (2.1.67) is an admissible test function for the V-problem. Indeed, we have

$$
\begin{aligned}
\Delta_2^{v^\varepsilon}(t) = \int_{\mathbb{R}^d\times\mathcal{Y}} & -\mathbb{D}_2^{\mathrm{ex}}(t)\nabla_y V^{\mathrm{ex}}(t) : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon(t)) \\
& + [\mathbb{F}_2^{\mathrm{ex}}(t, u^{\mathrm{ex}}(t), V^{\mathrm{ex}}(t)) - V_t^{\mathrm{ex}}(t)] \cdot \mathcal{T}_\varepsilon v^\varepsilon(t)\,\mathrm{d}x\,\mathrm{d}y \\
\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^d\times\mathcal{Y}} & -\mathbb{D}_2^{\mathrm{ex}}(t)\nabla_y V^{\mathrm{ex}}(t) : \nabla_y \widetilde{V}^{\mathrm{ex}}(t) \\
& + [\mathbb{F}_2^{\mathrm{ex}}(t, u^{\mathrm{ex}}(t), V^{\mathrm{ex}}(t)) - V_t^{\mathrm{ex}}(t)] \cdot \widetilde{V}^{\mathrm{ex}}(t)\,\mathrm{d}x\,\mathrm{d}y = 0,
\end{aligned}
$$

since V solves $(2.1.11.\mathrm{P}_0)_2$, see (2.1.48) for the weak formulation.

With the same arguments, we obtain L^∞-majorants depending on C_b, C_1, β for the approximation errors $\Delta_3^{v^\varepsilon}$ and $\Delta_4^{v^\varepsilon}$. Moreover, Lemma 1.2.6(a&b) and the assumptions in (2.1.14.Conv) on the convergence of the given data yield for all $t \in [0, T]$

$$
\begin{aligned}
|\Delta_3^{v^\varepsilon}(t)| &= \Big| \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathbb{D}_2^{\mathrm{ex}}(t) - \mathcal{T}_\varepsilon D_2^\varepsilon(t))\nabla_y V^{\mathrm{ex}}(t) : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t))\,\mathrm{d}x\,\mathrm{d}y \Big| \\
&\le 2C_b \|(\mathbb{D}_2^{\mathrm{ex}}(t) - \mathcal{T}_\varepsilon D_2^\varepsilon(t))\nabla_y V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} \xrightarrow{\varepsilon\to 0} 0
\end{aligned}
$$

as well as

$$
\begin{aligned}
&|\Delta_4^{v^\varepsilon}(t)| \\
&= \Big| \int_{\mathbb{R}^d\times\mathcal{Y}} [\mathcal{T}_\varepsilon F_2^\varepsilon(t, u^{\mathrm{ex}}(t), V^{\mathrm{ex}}(t)) - \mathbb{F}_2^{\mathrm{ex}}(t, u^{\mathrm{ex}}(t), V^{\mathrm{ex}}(t))] \cdot (\mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t))\,\mathrm{d}x\,\mathrm{d}y \Big| \\
&\le 2C_b \| \mathcal{T}_\varepsilon F_2^\varepsilon(t, u^{\mathrm{ex}}(t), V^{\mathrm{ex}}(t)) - \mathbb{F}_2^{\mathrm{ex}}(t, u^{\mathrm{ex}}(t), V^{\mathrm{ex}}(t))\|_{\mathbb{H}_{\mathbb{R}^d}} \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}
$$

Finally, $\Delta_5^{v^\varepsilon}(t) = L\| \mathcal{T}_\varepsilon u(t) - u^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}}^2 \to 0$ follows from Proposition 1.2.4(d).

Step 3: Controlling Δ^{u^ε}. We proceed as in Step 2. The first error term is bounded thanks to the growth condition on the given data (1.1.5) and (1.1.8) as well as the uniform bounds (2.1.10) and (2.1.13), i.e. $|\Delta_1^{u^\varepsilon}(t)| \le C(C_b, C_1, \beta)$, and, thus, we obtain with Proposition

1.2.9 for $\varepsilon \to 0$

$$
\begin{aligned}
&|\Delta_1^{u^\varepsilon}(t)| \\
&\leq C(C_b, C_1, \beta)\left(\|u(t) - \mathcal{G}_\varepsilon^0(u,U)(t)\|_H + \|\nabla\, \mathcal{G}_\varepsilon^0(u,U)(t) - \mathcal{F}_\varepsilon[\nabla u(t) + \nabla_y U(t)]^{\text{ex}}\|_H\right) \\
&\to 0.
\end{aligned}
$$

In the same manner, the second error is bounded by $|\Delta_2^{u^\varepsilon}(t)| \leq C(C_b, C_1, \beta)$. By the weak a priori convergence $\nabla u^\varepsilon \xrightarrow{2\text{w}} \nabla u + \nabla_y U$ in $\mathrm{L}^2(0,T;\mathbb{H})$, we obtain in the limit $\varepsilon \to 0$ an admissible pair of test functions (u, U) for equation (2.1.59) and hence for all $t \in [0,T]$:

$$
\begin{aligned}
\Delta_2^{u^\varepsilon}(t) = &\int_\Omega (F_{\text{eff}}(t, u(t), V(t)) - u_t(t)) \cdot u^\varepsilon(t)\,\mathrm{d}x \\
&- \int_{\mathbb{R}^d \times \mathcal{Y}} \mathbb{D}_1^{\text{ex}}(t)[\nabla u(t) + \nabla_y U(t)]^{\text{ex}} : \mathcal{T}_\varepsilon(\nabla u^\varepsilon(t))\,\mathrm{d}x\,\mathrm{d}y \to 0.
\end{aligned}
$$

The third error in (2.1.36) is bounded, too, and it vanishes for $\varepsilon \to 0$ by applying (2.1.14.Conv) and Lemma 1.2.6

$$
|\Delta_3^{u^\varepsilon}(t)| \leq 2C_b \|(\mathbb{D}_1^{\text{ex}}(t) - \mathcal{T}_\varepsilon\, D_1^\varepsilon(t))[\nabla u + \nabla_y U]^{\text{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} \to 0.
$$

Similarly, the last error terms in (2.1.38)–(2.1.40) are bounded and they vanish thanks to $u^\varepsilon \to u$ in (2.1.66), the Lipschitz continuity of F_1^ε, and Proposition 1.2.4(c)

$$
\begin{aligned}
|\Delta_4^{u^\varepsilon}(t)| &\leq C(C_b, C_1)\|u^\varepsilon(t) - u(t)\|_H \to 0, \\
|\Delta_5^{u^\varepsilon}(t)| &= L\|V^{\text{ex}}(t) - \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V^{\text{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}}^2 \to 0.
\end{aligned}
$$

In Step 2–3, we have shown that $\Delta^{w^\varepsilon}(t) \xrightarrow{\varepsilon\to 0} 0$ pointwise for all $t \in [0,T]$ with $w^\varepsilon \in \{u^\varepsilon, v^\varepsilon\}$. In particular, all error terms $|\Delta^{w^\varepsilon}(t)| \leq C(C_1, \beta, C_b)$ are bounded by the assumptions on the data and the uniform boundedness of the solutions. Hence, Lebesgue's dominated convergence theorem yields $\int_0^T \Delta^{v^\varepsilon}(t) + \Delta^{u^\varepsilon}(t)\,\mathrm{d}t \xrightarrow{\varepsilon\to 0} 0$. Moreover, the convergence of the initial values $v_0^\varepsilon \xrightarrow{2\text{s}} V_0$ and $u_0^\varepsilon \to u_0$ by (2.1.14.Conv) yields the limit in (2.0.5.Est):

$$
\max_{t\in[0,T]} \left\{\|\,\mathcal{T}_\varepsilon\, v^\varepsilon(t) - V^{\text{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(t) - u(t)\|_H\right\} \xrightarrow{\varepsilon\to 0} 0.
$$

In particular, we obtain the strong two-scale convergence of the subsequence extracted in (2.1.67). With the usual arguments, by considering another, different subsequence and since (u, V) is the unique weak solution of (2.1.11.P$_0$), we conclude that $v^\varepsilon(t) \xrightarrow{2\text{s}} V(t)$ in $\mathbb{H}$ uniformly for all $t \in [0,T]$, even for the *whole* sequence. Hence, (2.1.15a) is proved.

Step 4: Proof of the remaining two-scale convergences. Recalling $W^\varepsilon(t) := \mathcal{T}_\varepsilon\, v^\varepsilon(t) - V^{\text{ex}}(t)$, we obtain from (2.1.55) for all $t \in [0,T]$

$$
\begin{aligned}
&\alpha\|\nabla_y W^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}}^2 \leq \int_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon\, D_2^\varepsilon \nabla_y W^\varepsilon : \nabla_y W^\varepsilon\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_{\mathbb{R}^d\times\mathcal{Y}} -W_t^\varepsilon \cdot W^\varepsilon + [\mathcal{T}_\varepsilon\, F_2^\varepsilon(\mathcal{T}_\varepsilon\, u^\varepsilon, \mathcal{T}_\varepsilon\, v^\varepsilon) - \mathcal{T}_\varepsilon\, F_2^\varepsilon(u^{\text{ex}}, V^{\text{ex}})] \cdot W^\varepsilon\,\mathrm{d}x\,\mathrm{d}y + \Delta^{v^\varepsilon} \\
&\leq 2C_b\|W^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}} + L\left(\|\,\mathcal{T}_\varepsilon\, u^\varepsilon - u^{\text{ex}}\|_{\mathbb{H}} + \|W^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}}\right)\|W^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}} + \Delta^{v^\varepsilon} \to 0, \qquad (2.1.68)
\end{aligned}
$$

where we have used $\|W_t^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}} \leq \|v_t^\varepsilon\|_H + \|V_t\|_{\mathbb{H}} \leq 2C_b$, (2.1.15a), Proposition 1.2.4(d), and $\Delta^{v^\varepsilon} \to 0$ pointwise for all $t \in [0,T]$. Integrating (2.1.68) over $(0,T)$ yields $\nabla_y W^\varepsilon \to 0$ strongly in $\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})$ and hence (2.1.15b). Analogously, we rearrange equation (2.1.62) and obtain

$$\begin{aligned} &\alpha\| \mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\mathrm{ex}}\|^2_{\mathbb{H}_{\mathbb{R}^d}} \\ &\leq \tfrac{3}{2}L\left\{\|u^\varepsilon - u\|_H^2 + \|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|^2_{\mathbb{H}_{\mathbb{R}^d}}\right\} + \Delta^{u^\varepsilon} - \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 \to 0. \end{aligned} \tag{2.1.69}$$

Here, we used $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 = \int_\Omega (u^\varepsilon - u)_t \cdot (u^\varepsilon - u)\,\mathrm{d}x \leq 2C_b\|u^\varepsilon - u\|_H \to 0$. Again, integrating (2.1.69) over $(0,T)$ gives $\mathcal{T}_\varepsilon(\nabla u^\varepsilon) \to \nabla u + \nabla_y U$ strongly in $\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})$ and hence (2.1.15d). The remaining convergences in (2.1.15d) follow from Step 1.

Finally, we prove $v_t^\varepsilon \xrightharpoonup{2\mathrm{w}} V_t$ in $\mathbf{H} = \mathrm{L}^2(0,T;\mathbb{H})$. By the a priori bound (2.1.10) we know that $(\mathcal{T}_\varepsilon v^\varepsilon)_t$ is bounded in the Hilbert space $\mathbf{H}$, and hence has a weakly convergent subsequence with limit U. Choosing $\Phi \in C^\infty([0,T];\mathbb{H})$, the definition of the weak time derivative gives

$$(U,\Phi)_{\mathbf{H}} \xleftarrow{\varepsilon\to 0} (\mathcal{T}_\varepsilon(v_t^\varepsilon),\Phi)_{\mathbf{H}} = ((\mathcal{T}_\varepsilon v^\varepsilon)_t,\Phi)_{\mathbf{H}} = -(\mathcal{T}_\varepsilon v^\varepsilon,\Phi_t)_{\mathbf{H}} \xrightarrow{\varepsilon\to 0} -(V,\Phi_t)_{\mathbf{H}} = (V_t,\Phi)_{\mathbf{H}}.$$

Since Φ was arbitrary we conclude $U = V_t$ and (2.1.15c) is established. Thus, the proof of Theorem 2.1.1 is complete. □

2.2 Two-scale homogenization without improved time-regularity

In this section, we are going to relax the additional assumptions on the initial values, which guarantee improved time-regularity of the solutions for all times. In Theorem 2.2.1 below, we derive in principle the same homogenization result as in Theorem 2.1.1 without assuming (2.1.9.Time$_\varepsilon$) and (2.1.12.Time$_0$).

Theorem 2.2.1 (Main Theorem II)**.** *Let the assumptions* (2.1.9.Exist$_\varepsilon$)*,* (2.1.12.Exist$_0$)*, and* (2.1.14.Conv) *be satisfied. The sequence of solutions* $(u^\varepsilon, v^\varepsilon)_\varepsilon$ *of* (2.0.1.P$_\varepsilon$) *converges to the solution* (u,V) *of* (2.0.2.P$_0$) *in the following sense:*

$$\begin{aligned} &\textit{uniformly for all } t\in[0,T]: \ v^\varepsilon(t) \xrightarrow{2\mathrm{s}} V(t) \ \textit{in } \mathbb{H}, \\ &\textit{i.e.}\ \max\nolimits_{0\leq t\leq T}\|\mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} \to 0, \end{aligned} \tag{2.2.1a}$$

$$\varepsilon\nabla v^\varepsilon \xrightarrow{2\mathrm{s}} \nabla_y V \ \textit{in } \mathrm{L}^2(0,T;\mathbb{H}), \tag{2.2.1b}$$

$$\begin{aligned} &u^\varepsilon \rightharpoonup u \ \textit{in } \mathrm{L}^2(0,T;X) \ \textit{and } u_t^\varepsilon \rightharpoonup u_t \ \textit{in } \mathrm{L}^2(0,T;X^*), \\ &\textit{moreover, there exists a function } U\in\mathrm{L}^2(\Omega;\mathbb{X}_0) \textit{ such that} \\ &\nabla u^\varepsilon \xrightarrow{2\mathrm{s}} \nabla u + \nabla_y U \ \textit{in } \mathrm{L}^2(0,T;\mathbb{H}). \end{aligned} \tag{2.2.1c}$$

Due to the unimproved regularity of the solutions, we do not prove pointwise in time convergence of the gradients. Moreover, we cannot make any statement about the weak two-scale convergence of v_t^ε because the periodic unfolding operator $\mathcal{T}_\varepsilon$ is only well-defined for L^1-functions.

The idea of the proof of Main Theorem II is to approximate general L^2-initial values with a sequence of regularized initial values satisfying (2.1.9.Time$_\varepsilon$) and (2.1.12.Time$_0$).

Therefore, we study in Subsection 2.2.1 abstract reaction-diffusion systems and define a *regularization operator* $\mathcal{R}_\delta$ for initial values such that the associated regularized solutions are of improved time-regularity and converge strongly in $\mathrm{L}^2(0,T;X)$ to the original solution. In Subsection 2.2.2, we adjust the regularization to our multiscale problem (2.0.1.P$_\varepsilon$). Finally, we give the proof of Theorem 2.2.1 in Subsection 2.2.3. We complete this section with a discussion on the satisfiability of the assumptions for Main Theorem I and II in Subsection 2.2.4.

2.2.1 Regularization of general systems

In this subsection, we consider arbitrary Hilbert spaces $X \subset H$ as in Section 1.1. Independently of $\varepsilon > 0$, we consider the following general reaction-diffusion system

$$\begin{aligned} u_t &= \mathcal{A}u + F(u) &&\text{in } [0,T]\times\Omega,\\ u(0) &= u_0^{(\delta)} &&\text{in } \Omega. \end{aligned} \tag{2.2.2.P$_\delta$}$$

We recall $\mathcal{A}(t,x)u = \operatorname{div}(D(t,x)\nabla u)$ provided with homogeneous Neumann boundary conditions, i.e. $(D\nabla u)\cdot\nu = 0$ on $[0,T]\times\partial\Omega$. Moreover, we assume $D \in \mathcal{M}(\Omega)$ and $F \in \mathcal{F}(\Omega)$ so that solutions exist (according to Theorem 1.1.2).

For given $u_0 \in H$, we call a solution $u \in W(0,T;X)$ of (2.2.2.P$_\delta$) with $u(0) = u_0$ *general solution.* And for more regular initial values $u_0^\delta \in H$ with $\mathcal{A}(0)u_0^\delta \in H$, we call the associated solution $u^\delta \in W_{\mathrm{imp}}(0,T;X)$ of (2.2.2.P$_\delta$) *regularized solution*, i.e. u^δ is of improved time-regularity by Proposition 1.1.3. In the following, we construct a regularization operator $\mathcal{R}_\delta$ via standard elliptic regularization.

Definition 2.2.2. *For $\delta > 0$, the operator $\mathcal{R}_\delta : H \to X_\delta$ maps $u \in H$ to $u^\delta := \mathcal{R}_\delta\, u \in X$, where u^δ is the unique solution of the elliptic problem*

$$u^\delta - \operatorname{div}(\delta^2 D(0)\nabla u^\delta) = u \ \textit{in}\ \Omega, \quad (D(0)\nabla u^\delta)\cdot\nu = 0\ \textit{on}\ \partial\Omega. \tag{2.2.3}$$

Here, the space X_δ is defined as X_ε in (1.2.14). By construction, we obtain

$$\text{for all } \delta > 0 \text{ and } u \in H: \quad \mathcal{A}(0)(\mathcal{R}_\delta u) \in H. \tag{2.2.4}$$

The weak formulation of (2.2.3) reads

$$\int_\Omega (u^\delta - u)\cdot\varphi + D(0)\delta\nabla u^\delta : \delta\nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in X. \tag{2.2.5}$$

Testing (2.2.5) with $\varphi = u^\delta$ yields the uniform bound

$$\|\,\mathcal{R}_\delta\, u\|_{X_\delta} := \|u^\delta\|_H + \delta\|\nabla u^\delta\|_H \le C_\alpha \|u\|_H \quad \text{with} \quad C_\alpha := \tfrac{1}{\min\{1,\alpha\}}, \tag{2.2.6}$$

where $\alpha > 0$ is the lower bound for the ellipticity of D.

In the following two lemmata, we show that $\mathcal{R}_\delta u_0$ recovers any given initial value u_0 strongly, i.e. $\|\mathcal{R}_\delta u_0 - u_0\|_H \to 0$ and, moreover, that the associated regularized solution u^δ converges to the general solution u.

Lemma 2.2.3. *For the regularization operator $\mathcal{R}_\delta$ as in Definition* (2.2.2)*, we have*

$$\textit{for all } u \in H: \quad \mathcal{R}_\delta\, u \to u \quad \textit{in } H. \tag{2.2.7}$$

Proof. For arbitrary $u \in H$, we set $u^\delta := \mathcal{R}_\delta\, u$. From the boundedness (2.2.6), we obtain the existence of limit functions $w, z \in H$ and a subsequence δ' of δ so that $u^{\delta'} \rightharpoonup w$ and $\delta'\nabla u^{\delta'} \rightharpoonup z$ in H. For arbitrary $\varphi \in X$ fixed, it holds $\delta\nabla\varphi \to 0$ in H, and therefore, passing to the limit $\delta \to 0$ in (2.2.5) yields $\int_\Omega (w-u)\cdot\varphi + D(0)z : 0\,\mathrm{d}x = 0$ for all $\varphi \in X$. The fundamental lemma in the calculus of variations implies $w = u$ a.e. in Ω.

Choosing another subsequence with $u^{\delta''} \rightharpoonup w^*$ and $\delta''\nabla u^{\delta''} \rightharpoonup z^*$ in H and passing again to the limit $\delta \to 0$ in (2.2.5) gives $w^* = u$ a.e. in Ω. Thus the whole sequence converges, i.e. $u^\delta \rightharpoonup u$ in H.

Since H is reflexive, it remains to show $\|u^\delta\|_H \to \|u\|_H$ and the strong convergence follows immediately. On the one hand, the weak lower semicontinuity of the norm gives $\liminf_{\delta\to0}\|u^\delta\|_H^2 \geq \|u\|_H^2$. On the other hand, the ellipticity of D yields

$$\lim_{\delta\to0}\|u^\delta\|_H^2 = \lim_{\delta\to0}\int_\Omega u\cdot u^\delta \underbrace{-D(0)\delta\nabla u^\delta : \delta\nabla u^\delta}_{\leq -\alpha|\delta\nabla u^\delta|^2 \leq 0}\,\mathrm{d}x \leq \lim_{\delta\to0}\int_\Omega u\cdot u^\delta\,\mathrm{d}x = \|u\|_H^2.$$

Combining the lower and upper estimate gives the desired strong convergence. □

Lemma 2.2.4. *For given $u_0 \in H$, let u denote the* general solution *of* (2.2.2.P$_\delta$) *with $u(0) = u_0$ and let u^δ denote the associated* regularized solution *with $u^\delta(0) = \mathcal{R}_\delta\, u_0$. Then, there exists a constant $C \geq 0$ only depending on the parameters α, L, and T such that*

$$\|u^\delta - u\|_{C([0,T];H)} + \|\nabla u^\delta - \nabla u\|_{\mathrm{L}^2(0,T;H)} \leq C\|\,\mathcal{R}_\delta\, u_0 - u_0\|_H.$$

Proof. Taking the difference of the weak formulations of $u_t = \mathcal{A}u + F(u)$ and $u_t^\delta = \mathcal{A}u^\delta + F(u^\delta)$ as well as testing with $\varphi = u^\delta - u$, we obtain for all $t \in [0,T]$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\delta - u\|_H^2 &= \int_\Omega -D\nabla(u^\delta - u) : \nabla(u^\delta - u) + [F(u^\delta) - F(u)]\cdot(u^\delta - u)\,\mathrm{d}x \\ &\leq -\alpha\|\nabla(u^\delta - u)\|_H^2 + L\|u^\delta - u\|_H^2. \end{aligned} \tag{2.2.8}$$

Applying Gronwall's lemma yields $\|u^\delta(t) - u(t)\|_H^2 \leq e^{2LT}\|\,\mathcal{R}_\delta\, u_0 - u_0\|_H^2$ for all $t \in [0,T]$ and hence $\|u^\delta - u\|_{C([0,T];H)} \leq C\|\,\mathcal{R}_\delta\, u_0 - u_0\|_H$. Rearranging (2.2.8) and integrating over $[0,T]$ gives $\alpha\|\nabla(u^\delta - u)\|_{\mathrm{L}^2(0,T;H)} \leq C\|u^\delta - u\|_{C([0,T];H)}$, which finishes the proof. □

2.2.2 Regularization and homogenization of the coupled system

Based on the observations in the previous subsection, we combine the homogenization result stated in Theorem 2.1.1 and the regularization of L^2-initial values. Therefore, let the spaces $X \subset H$ and $\mathbb{X} \subset \mathbb{H}$ be as in (2.0.6). As in Subsection 2.1.2, we write the coupled system (2.0.1.P$_\varepsilon$) for $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ shortly in the abstract form

$$w_t^\varepsilon = \mathcal{A}_\varepsilon w^\varepsilon + F^\varepsilon(w^\varepsilon) \quad \text{in } \Omega \qquad \text{with} \qquad w^\varepsilon(0) = w_0^\varepsilon, \tag{2.2.9.P$_{\varepsilon,0}$}$$

where $\mathcal{A}_\varepsilon w = \mathrm{div}(D^\varepsilon\nabla w)$ provided with homogeneous Neumann boundary conditions and

$$D^\varepsilon = \begin{pmatrix} D_1^\varepsilon & 0 \\ 0 & \varepsilon^2 D_2^\varepsilon \end{pmatrix} \qquad \text{and} \qquad F^\varepsilon(w) = \begin{pmatrix} F_1^\varepsilon(u,v) \\ F_2^\varepsilon(u,v) \end{pmatrix}.$$

Let the assumptions (2.1.9.Exist$_\varepsilon$) and (2.1.12.Exist$_0$) hold true throughout this chapter. Since $w_0^\varepsilon = (u_0^\varepsilon, v_0^\varepsilon) \in H$ only, we obtain according to Theorem 1.1.2 the existence of a *general solution* $w^\varepsilon \in W(0,T;X)$. Due to the unimproved regularity of the solution w^ε, we cannot apply Theorem 2.1.1 directly. Therefore, we construct a suitably regularized initial value $w_0^{\varepsilon,\delta}$ so that the assumptions (2.1.9.Time$_\varepsilon$) for improved time-regularity (Proposition 1.1.3) are satisfied. Such a *regularized solution* $w^{\varepsilon,\delta} \in W_{\text{imp}}(0,T;X)$ solves the system

$$w_t^{\varepsilon,\delta} = \mathcal{A}_\varepsilon w^{\varepsilon,\delta} + F^\varepsilon(w^{\varepsilon,\delta}) \quad \text{in } \Omega \qquad \text{with} \qquad w^{\varepsilon,\delta}(0) = w_0^{\varepsilon,\delta}. \tag{2.2.9.P$_{\varepsilon,\delta}$}$$

In order to pass to the limit $\varepsilon \to 0$ in (2.2.9.P$_{\varepsilon,0}$) respective (2.2.9.P$_{\varepsilon,\delta}$), we further assume (2.1.14.Conv). Intuitively, we choose $w_0^{\varepsilon,\delta} := (\mathcal{R}_\delta\, u_0^\varepsilon, \mathcal{R}_\delta\, v_0^\varepsilon)$ as in Definition 2.2.2 and obtain the strong convergences

$$\mathcal{R}_\delta\, u_0^\varepsilon \xrightarrow[\delta\to 0]{} u_0^\varepsilon \xrightarrow[\varepsilon\to 0]{} u_0 \quad \text{respective} \quad \mathcal{R}_\delta\, v_0^\varepsilon \xrightarrow[\delta\to 0]{} v_0^\varepsilon \xrightarrow[\varepsilon\to 0]{2s} V_0 \tag{2.2.11}$$

by the application of Lemma 2.2.3 and the assumptions $u_0^\varepsilon \to u_0$ respective $v_0^\varepsilon \xrightarrow{2\text{s}} V_0$, successively. Here, $W_0 = (u_0, V_0)$ denotes the initial value for the general solution $W = (u, V)$ of the limit problem

$$W_t = \mathbb{A}W + \mathbb{F}(W) \quad \text{in } \Omega \times \mathcal{Y} \qquad \text{with} \qquad W(0) = W_0, \tag{2.2.12.P$_{0,0}$}$$

where $u \in W(0,T;X)$ and $V \in W(0,T;\mathbb{X})$. We recall

$$\mathbb{A}W = \begin{pmatrix} \operatorname{div}(D_{\text{eff}}\nabla u) \\ \operatorname{div}_y(\mathbb{D}_2\nabla_y V) \end{pmatrix} \qquad \text{and} \qquad \mathbb{F}(W) = \begin{pmatrix} F_{\text{eff}}(u,V) \\ \mathbb{F}_2(u,V) \end{pmatrix}.$$

Returning to (2.2.11), we observe that assumption (2.1.9.Time$_\varepsilon$) fails in general, i.e. $\mathcal{A}_\varepsilon(0)w_0^{\varepsilon,\delta} \notin H$, since $D \neq D_i^\varepsilon$. Therefore, we have to adjust the operator $\mathcal{R}_\delta$ to the ε-microstructure of problem (2.2.9.P$_{\varepsilon,0}$). Moreover, we need that the convergences in (2.2.11) commute in the sense that letting first $\varepsilon \to 0$ and afterward $\delta \to 0$ gives the same limit. In the following, we construct regularization operators $\widetilde{\mathcal{R}}_\delta, \mathcal{R}_\delta{}^*$ such that

$$\widetilde{\mathcal{R}}_\delta w_0^\varepsilon \xrightarrow[\varepsilon\to 0]{!} \mathcal{R}_\delta{}^*\, W_0 \xrightarrow[\delta\to 0]{} W_0 \quad \text{with} \quad \mathcal{A}_\varepsilon(0)(\widetilde{\mathcal{R}}_\delta w_0^\varepsilon) \overset{!}{\in} H \text{ and } \mathbb{A}(0)(\mathcal{R}_\delta{}^*\, W_0) \overset{!}{\in} \mathbb{H}. \tag{2.2.13}$$

Identifying $\mathcal{R}_\delta{}^*\, W_0$ with W_0^δ, we denote with $W^\delta = (u^\delta, V^\delta)$, where $u^\delta \in W_{\text{imp}}(0,T;X)$ and $V^\delta \in W_{\text{imp}}(0,T;\mathbb{X})$, the associated regularized solution of W, i.e.

$$W_t^\delta = \mathbb{A}W^\delta + \mathbb{F}(W^\delta) \quad \text{in } \Omega \times \mathcal{Y} \qquad \text{with} \qquad W^\delta(0) = W_0^\delta. \tag{2.2.14.P$_{0,\delta}$}$$

What is a meaningful choice for $\widetilde{\mathcal{R}}_\delta$ and $\mathcal{R}_\delta{}^*$ in (2.2.13)? Based on Definition 2.2.2, we define for $\varepsilon, \delta > 0$ the following regularization operators

$$\mathcal{R}_{\varepsilon,\delta}^0 : H \to X_\delta;\ u \mapsto u^{\varepsilon,\delta} \text{ with } u^{\varepsilon,\delta} - \operatorname{div}(\delta^2 D_1^\varepsilon(0)\nabla u^{\varepsilon,\delta}) = u \text{ in } \Omega; \tag{2.2.15a}$$

$$\mathcal{R}_{\varepsilon,\delta}^1 : H \to X_{\varepsilon\delta};\ v \mapsto v^{\varepsilon,\delta} \text{ with } v^{\varepsilon,\delta} - \operatorname{div}(\delta^2\varepsilon^2 D_2^\varepsilon(0)\nabla v^{\varepsilon,\delta}) = v \text{ in } \Omega; \tag{2.2.15b}$$

$$\mathcal{R}_\delta : H \to X_\delta;\ u \mapsto u^\delta \text{ with } u^\delta - \operatorname{div}(\delta^2 D_{\text{eff}}(0)\nabla u^\delta) = u \text{ in } \Omega; \tag{2.2.15c}$$

$$\mathbb{R}_\delta : \mathbb{H} \to \mathbb{X}_\delta;\ V \mapsto V^\delta \text{ with } V^\delta - \operatorname{div}(\delta^2\mathbb{D}_2(0)\delta\nabla_y V^\delta) = V \text{ in } \Omega \times \mathcal{Y}. \tag{2.2.15d}$$

We supplement (2.2.15a)–(2.2.15c) with homogeneous Neumann boundary conditions on $\partial\Omega$ and point out that $\mathcal{Y}$ implies periodic boundary conditions in (2.2.15d). Finally, we choose the regularized initial values in (2.2.9.P$_{\varepsilon,\delta}$) and (2.2.14.P$_{0,\delta}$) as

$$w_0^{\varepsilon,\delta} := (\mathcal{R}^0_{\varepsilon,\delta} u_0^\varepsilon, \mathcal{R}^1_{\varepsilon,\delta} v_0^\varepsilon) \quad \text{and} \quad W_0^\delta := (\mathcal{R}_\delta\, u_0, \mathbb{R}_\delta V_0). \tag{2.2.16}$$

According to (2.2.4), the assumptions (2.1.9.Time$_\varepsilon$) and (2.1.12.Time$_0$) are satisfied, i.e. $\mathcal{A}_\varepsilon(0) w_0^{\varepsilon,\delta} \in H$ and $\mathbb{A}(0) W_0^\delta \in \mathbb{H}$. Therefore, the regularized solutions $w^{\varepsilon,\delta}$ and W^δ are indeed of improved time-regularity. It remains to verify the first convergence in (2.2.13) for $\varepsilon \to 0$.

Lemma 2.2.5. *For $i \in \{1,2\}$, let $\mathcal{T}_\varepsilon\, D_i^\varepsilon(t,x,y)$ converge pointwise to $\mathbb{D}_i^{\mathrm{ex}}(t,x,y)$ for a.a. $(x,y) \in \mathbb{R}^d \times \mathcal{Y}$ and all $t \in [0,T]$. Let $\delta > 0$ be fixed.*

(a) For given sequences $u^\varepsilon \to u$ in H, we have $\mathcal{R}^0_{\varepsilon,\delta} u^\varepsilon \to \mathcal{R}_\delta\, u$ in H for $\varepsilon \to 0$.

(b) For given sequences $v^\varepsilon \xrightarrow{2\mathrm{s}} V$ in $\mathbb{H}$, we have $\mathcal{R}^1_{\varepsilon,\delta} v^\varepsilon \xrightarrow{2\mathrm{s}} \mathbb{R}_\delta V$ in $\mathbb{H}$ for $\varepsilon \to 0$.

Indeed, one also obtains strong two-scale convergence for the gradient terms, see e.g. Proposition 2.2.7 for $\varepsilon\nabla v^\varepsilon \xrightarrow{2\mathrm{s}} \nabla_y V$ in $\mathbb{H}$.

Proof of Lemma 2.2.5. The basic idea of the proof follows along the lines of the proof of Lemma 2.2.3, i.e. we deduce the weak convergence of a subsequence and improve it to strong convergence by showing the convergence of the norms.

Ad (a). For fixed $\delta > 0$, we set $u^{\varepsilon,\delta} := \mathcal{R}^0_{\varepsilon,\delta} u^\varepsilon \in X$. According to (2.2.6), we obtain the uniform bound $\|u^{\varepsilon,\delta}\|_{X_\delta} \le C_\alpha$ which implies the existence of a subsequence (not relabeled) and limit functions $u^\delta \in X$ and $U^\delta \in \mathbb{X}_0$ such that $u^{\varepsilon,\delta} \rightharpoonup u^\delta$ in X and $\nabla u^{\varepsilon,\delta} \xrightarrow{2\mathrm{w}} \nabla u^\delta + \nabla_y U^\delta$ in $\mathbb{H}$ for $\varepsilon \to 0$.

For arbitrary $(\varphi, \Phi) \in X \times \mathbb{X}_0$, we choose the test function $\varphi^\varepsilon = \mathcal{G}^0_\varepsilon(\varphi,\Phi) \in X$ in the weak formulation of (2.2.15a), cf. (2.2.5). The integral identity (1.2.7), Lemma 1.2.6(a) and Proposition 1.2.8(a), the assumption $u^\varepsilon \to u$ in H, and the a priori weak two-scale convergence give

$$\begin{aligned}
0 &= \int_\Omega (u^{\varepsilon,\delta} - u^\varepsilon)\cdot \varphi^\varepsilon \,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathcal{Y}} \delta^2\, \mathcal{T}_\varepsilon\, D_1^\varepsilon\, \mathcal{T}_\varepsilon(\nabla u^{\varepsilon,\delta}) : \mathcal{T}_\varepsilon(\nabla\varphi^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\
&\xrightarrow{\varepsilon\to 0} \int_\Omega (u^\delta - u)\cdot\varphi\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathcal{Y}} \delta^2 \mathbb{D}_1^{\mathrm{ex}} [\nabla u^\delta + \nabla_y U^\delta]^{\mathrm{ex}} : [\nabla\varphi + \nabla_y\Phi]^{\mathrm{ex}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_\Omega (u^\delta - u)\cdot\varphi + \delta^2 D_{\mathrm{eff}} \nabla u^\delta : \nabla\varphi\,\mathrm{d}x \\
&\iff u^\delta \text{ solves (2.2.15c)} \iff u^\delta = \mathcal{R}_\delta\, u.
\end{aligned}$$

Since H is reflexive, it remains to show $\|u^{\varepsilon,\delta}\|_H \to \|u^\delta\|_H$ and the strong convergence follows immediately. On the one hand, the weak lower semicontinuity of the norm gives $\liminf_{\varepsilon\to 0} \|u^{\varepsilon,\delta}\|_H^2 \ge \|u^\delta\|_H^2$. On the other hand, the ellipticity of D_1^ε yields

$$\lim_{\varepsilon\to 0} \|u^{\varepsilon,\delta}\|_H^2 = \lim_{\varepsilon\to 0} \int_\Omega u^\delta\cdot u^{\varepsilon,\delta} \underbrace{-\delta^2 D_1^\varepsilon \nabla u^{\varepsilon,\delta} : \nabla u^{\varepsilon,\delta}}_{\le -\alpha|\delta\nabla u^{\varepsilon,\delta}|^2 \le 0} \,\mathrm{d}x \le \lim_{\varepsilon\to 0}\int_\Omega u^\delta\cdot u^{\varepsilon,\delta}\,\mathrm{d}x = \|u^\delta\|_H^2.$$

Combining the lower and upper estimate gives the desired strong convergence in (a).

Ad (b). We proceed as in (a). For fixed $\delta > 0$, we set $v^{\varepsilon,\delta} := \mathcal{R}^1_{\varepsilon,\delta} v^\varepsilon \in X_\varepsilon$. According to (2.2.6), we obtain the uniform bound $\|v^{\varepsilon,\delta}\|_{X_{\varepsilon\delta}} \leq C_\alpha$ which implies the existence of a subsequence (not relabeled) and a limit function $V^\delta \in \mathbb{X}$ such that $v^{\varepsilon,\delta} \xrightharpoonup{2\mathrm{w}} V^\delta$ and $\varepsilon\nabla v^{\varepsilon,\delta} \xrightharpoonup{2\mathrm{w}} \nabla_y V^\delta$ in $\mathbb{H}$ for $\varepsilon \to 0$.

For arbitrary $\Phi \in \mathbb{X}$, we test (2.2.15b) with $\varphi^\varepsilon = \mathcal{G}^1_\varepsilon \Phi \in X_\varepsilon$. The integral identity, Lemma 1.2.6(a), Proposition 1.2.8(b), the assumption $v^\varepsilon \xrightarrow{2\mathrm{s}} V$ in $\mathbb{H}$, and the a priori weak two-scale convergence give

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon(v^{\varepsilon,\delta} - v^\varepsilon) \cdot \mathcal{T}_\varepsilon \varphi^\varepsilon + \delta^2\, \mathcal{T}_\varepsilon D^\varepsilon_2\, \mathcal{T}_\varepsilon(\varepsilon\nabla v^{\varepsilon,\delta}) : \mathcal{T}_\varepsilon(\varepsilon\nabla\varphi^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\
&\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^d\times\mathcal{Y}} (V^{\delta,\mathrm{ex}} - V^{\mathrm{ex}}) \cdot \Phi^{\mathrm{ex}} + \delta^2 \mathbb{D}^{\mathrm{ex}}_2 \nabla_y V^{\delta,\mathrm{ex}} : \nabla_y \Phi^{\mathrm{ex}}\,\mathrm{d}x\,\mathrm{d}y \\
&\iff V^\delta \text{ solves (2.2.15d)} \iff V^\delta = \mathbb{R}_\delta V.
\end{aligned}
$$

Since $\mathbb{H}$ is reflexive, the strong convergence follows from $\|\mathcal{T}_\varepsilon v^{\varepsilon,\delta}\|_{\mathbb{H}} \to \|V^\delta\|_{\mathbb{H}}$ as in (a). □

2.2.3 Proof of Main Theorem II

Relying on the results of the previous subsections, we can now prove Theorem 2.2.1, which states the two-scale homogenization of the coupled system without asking the initial values to be well-prepared, i.e. without using improved time-regularity of the solutions. Owing to Subsection 2.2.2, we can approximate any general solution with a regularized solution. The Theorems 2.1.6 and 2.1.1 are then applicable to the regularized solutions.

Proof of Theorem 2.2.1. For given initial values $u^\varepsilon_0, v^\varepsilon_0, u_0 \in H$, and $V_0 \in \mathbb{H}$, we denote with $u^\varepsilon \in W(0,T;X)$, $v^\varepsilon \in W(0,T;X_\varepsilon)$ and $u \in W(0,T;X)$, $V \in W(0,T;\mathbb{X})$ the solutions of (2.0.1.P$_\varepsilon$) and (2.0.2.P$_0$), respectively. We regularize their initial values $(u^\varepsilon_0, v^\varepsilon_0)$ and (u_0, V_0) as in (2.2.16) using the regularization operators (2.2.15a)–(2.2.15d), i.e.

$$
u^{\varepsilon,\delta}_0 = \mathcal{R}^0_{\varepsilon,\delta} u^\varepsilon_0, \quad v^{\varepsilon,\delta}_0 = \mathcal{R}^1_{\varepsilon,\delta} v^\varepsilon_0, \quad u^\delta_0 = \mathcal{R}_\delta u_0, \quad \text{and} \quad V^\delta_0 = \mathbb{R}_\delta V_0.
$$

With this, the associated regularized solutions are denoted by $u^{\varepsilon,\delta} \in W_{\mathrm{imp}}(0,T;X)$, $v^{\varepsilon,\delta} \in W_{\mathrm{imp}}(0,T;X_\varepsilon)$ and $u^\delta \in W_{\mathrm{imp}}(0,T;X)$, $V^\delta \in W_{\mathrm{imp}}(0,T;\mathbb{X})$.

In the following, we apply

1. the triangle inequality,
2. Lemma 2.2.4 to the difference of general and regularized solutions, and
3. estimate (2.0.5.Est) to the difference of two regularized solutions:

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\max_{0\leq t\leq T} \{\|\mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(t) - u(t)\|_H\}\Big) \\
&\overset{1.}{\leq} \lim_{\varepsilon\to 0}\Big(\max_{0\leq t\leq T} \{\|v^\varepsilon(t) - v^{\varepsilon,\delta}(t)\|_H + \|\mathcal{T}_\varepsilon v^{\varepsilon,\delta}(t) - V^{\delta,\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} + \|V^{\delta,\mathrm{ex}}(t) - V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} \\
&\qquad + \|u^\varepsilon(t) - u^{\varepsilon,\delta}(t)\|_H + \|u^{\varepsilon,\delta}(t) - u^\delta(t)\|_H + \|u^\delta(t) - u(t)\|_H\}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{2.}{\leq} \lim_{\varepsilon\to 0}\Big(C\{\|v_0^\varepsilon - v_0^{\varepsilon,\delta}\|_H + \|V_0^\delta - V_0\|_{\mathbb{H}} + \|u_0^\varepsilon - u_0^{\varepsilon,\delta}\|_H + \|u_0^\delta - u_0\|_H\}\\
&\quad + \max_{0\le t\le T}\{\|\mathcal{T}_\varepsilon v^{\varepsilon,\delta}(t) - V^{\delta,\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u^{\varepsilon,\delta}(t) - u^\delta(t)\|_H\}\Big)\\
&\overset{3.}{\leq} \lim_{\varepsilon\to 0} C\Big\{\|v_0^\varepsilon - v_0^{\varepsilon,\delta}\|_H + \|V_0^\delta - V_0\|_{\mathbb{H}} + \|u_0^\varepsilon - u_0^{\varepsilon,\delta}\|_H + \|u_0^\delta - u_0\|_H\\
&\qquad\qquad + \|\mathcal{T}_\varepsilon v_0^{\varepsilon,\delta} - V_0^{\delta,\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u_0^{\varepsilon,\delta} - u_0^\delta\|_H + \int_0^T \Delta^{v^{\varepsilon,\delta}}(t) + \Delta^{u^{\varepsilon,\delta}}(t)\,\mathrm{d}t\Big\}\\
&= C\{\|V_0 - V_0^\delta\|_{\mathbb{H}} + \|u_0 - u_0^\delta\|_H + 0\},
\end{aligned}
\tag{2.2.17}
$$

where $C \geq 0$ is a generic constant depending on $L, T > 0$ as in Theorem 2.1.6. The last equality (i.e. $\varepsilon \to 0$) follows from the strong (two-scale) convergence of the initial values as well as the commutativity of the limits, see Lemma 2.2.5, i.e.

$$
\begin{aligned}
&\|u_0^\varepsilon - u_0^{\varepsilon,\delta}\|_H \xrightarrow{\varepsilon\to 0} \|u_0 - u_0^\delta\|_H \text{ and } \|u_0^{\varepsilon,\delta} - u_0^\delta\|_H \xrightarrow{\varepsilon\to 0} 0,\\
&\|v_0^\varepsilon - v_0^{\varepsilon,\delta}\|_H \xrightarrow{\varepsilon\to 0} \|V_0 - V_0^\delta\|_{\mathbb{H}} \text{ and } \|\mathcal{T}_\varepsilon v_0^{e,\delta} - V_0^{\delta,\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}
$$

Moreover, we used the convergence of the error terms $\int_0^T \Delta^{v^{\varepsilon,\delta}} + \Delta^{u^{\varepsilon,\delta}}\,\mathrm{d}t \to 0$ as $\varepsilon \to 0$ for each $\delta > 0$, see Subsection 2.1.6. Thanks to Lemma 2.2.3, we can pass to the limit $\delta \to 0$ in (2.2.17) and arrive at

$$
\lim_{\varepsilon\to 0}\Big(\max_{0\le t\le T}\Big\{\|\mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u^\varepsilon(t) - u(t)\|_H\Big\}\Big) = 0,
\tag{2.2.18}
$$

which proves (2.2.1a).

We continue by deriving the strong two-scale convergence of the gradients (2.2.1b)–(2.2.1c), i.e. $\varepsilon\nabla v^\varepsilon \xrightarrow{2\mathrm{s}} \nabla_y V$ in $\mathrm{L}^2(0,T;\mathbb{H})$ and $\nabla u^\varepsilon \xrightarrow{2\mathrm{s}} \nabla u + \nabla_y U$ in $\mathrm{L}^2(0,T;\mathbb{H})$. We only show the gradient estimate for v^ε and the one for u^ε follows analogously.
As before, we apply

1. the triangle inequality,
2. Lemma 2.2.4,
3. estimate (2.1.68),
4. Lemma 2.2.5 and $v_0^\varepsilon \xrightarrow{2\mathrm{s}} V_0$ and (2.2.18), as well as
5. Lemma 2.2.3 (by choosing $\mathbb{R}_\delta$ and $\mathbb{H}$ as in Subsection 2.2.2 for the abstract $\mathcal{R}$ and H in Subsection 2.2.1):

$$
\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\Big(\|\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon) - \nabla_y V^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}\Big)\\
&\overset{1.}{\leq} \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\Big(\|\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon) - \mathcal{T}_\varepsilon(\varepsilon\nabla v^{\varepsilon,\delta})\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})} + \|\mathcal{T}_\varepsilon(\varepsilon\nabla v^{\varepsilon,\delta}) - \nabla_y V^{\delta,\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}\\
&\qquad\qquad + \|\nabla_y V^\delta - \nabla_y V\|_{\mathrm{L}^2(0,T;\mathbb{H})}\Big)\\
&\overset{2.}{\leq} \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\Big(C\Big\{\|v_0^\varepsilon - v_0^{\varepsilon,\delta}\|_H + \|V_0^\delta - V_0\|_{\mathbb{H}}\Big\} + \|\mathcal{T}_\varepsilon(\varepsilon\nabla v^{\varepsilon,\delta}) - \nabla_y V^{\delta,\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}\Big)
\end{aligned}
$$

$$\overset{3.}{\leq} \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\Bigg(C\Bigg\{\|v_0^\varepsilon - v_0^{\varepsilon,\delta}\|_H + \|V_0^\delta - V_0\|_{\mathbb{H}}$$
$$+\Bigg(\int_0^T C_b\|\mathcal{T}_\varepsilon v^{\varepsilon,\delta}(t) - V^{\delta,\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}} + \Delta^{v^{\varepsilon,\delta}}(t)$$
$$+ L\left(\|\mathcal{T}_\varepsilon v^{\varepsilon,\delta}(t) - V^{\delta,\mathrm{ex}}(t)\|^2_{\mathbb{H}_{\mathbb{R}^d}} + \|\mathcal{T}_\varepsilon u^{\varepsilon,\delta}(t) - u^\delta(t)\|^2_H\right)\mathrm{d}t\Bigg)^{1/2}\Bigg\}\Bigg)$$
$$\overset{4.}{=} \lim_{\delta\to 0} C\{\|V_0^\delta - V_0\|_{\mathbb{H}} + 0\} \overset{5.}{=} 0.$$

The remaining convergence $u^\varepsilon \rightharpoonup u$ in $W(0,T;X)$ follows from the a priori boundedness (cf. Step 1 in the Proof of Theorem 2.1.1). □

2.2.4 Discussion of the assumptions

We discuss different choices for the given data such that the assumptions of Main Theorem I and II are satisfied. For $i \in \{1,2\}$, let the data $\mathbb{D}_i \in \mathcal{M}(\Omega\times\mathcal{Y})$, $\mathbb{F}_i \in \mathcal{F}(\Omega\times\mathcal{Y})$ and $V_0 \in \mathbb{H}$ be given. The natural choice for v_0^ε and F_i^ε is to set

$$v_0^\varepsilon := \mathcal{F}_\varepsilon V_0^{\mathrm{ex}} \quad \text{and} \quad F_i^\varepsilon(t,\cdot,A,B) := \mathcal{F}_\varepsilon \mathbb{F}_i^{\mathrm{ex}}(t,\cdot,\cdot,A,B) \tag{2.2.19}$$

for all $(t,A,B) \in [0,T]\times\mathbb{R}^{m_1+m_2}$. Clearly, we have $v_0^\varepsilon \in H$ and $F_i^\varepsilon \in \mathcal{F}(\Omega)$. Moreover, Proposition 1.2.4(c) implies $v_0^\varepsilon \xrightarrow{2s} V_0$ in $\mathbb{H}$ as well as $F_i^\varepsilon(t,\cdot,A,B) \xrightarrow{2s} \mathbb{F}_i(t,\cdot,\cdot,A,B)$ in $\mathbb{H}$ for all $(t,A,B) \in [0,T]\times\mathbb{R}^{m_1+m_2}$. In the case of classical diffusion, we can choose $u_0^\varepsilon \equiv u_0 \in H$, whereas such a choice is not admissible in the case of slow diffusion due to the "true" two-scale character of the effective model.

In order to guarantee the uniform ellipticity of D_i^ε, we set

$$D_i^\varepsilon(t,x) := \begin{cases} \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} \mathbb{D}_i(t,\{x/\varepsilon\}_Y, z)\,\mathrm{d}z & \text{if } (x,y)\in\Omega_\varepsilon^-\times\mathcal{Y}, \\ \alpha\mathbb{I} & \text{if } (x,y)\in\Omega\backslash\Omega_\varepsilon^-\times\mathcal{Y}. \end{cases} \tag{2.2.20}$$

Here, $\mathbb{I}$ denotes the identity tensor in $\mathbb{R}^{(m\times d)\times(m\times d)}$. We point out that the boundedness (1.1.5) for $\mathbb{D}_i$ is imposed *for all* (x,y) and, thus, it holds $D_i^\varepsilon \in \mathrm{L}^\infty(\Omega)$. Therefore, we have $D_i^\varepsilon \in \mathcal{M}(\Omega)$ and for all $t\in[0,T]$: $\mathcal{T}_\varepsilon D_i^\varepsilon(t,x,y) \to \mathbb{D}_i^{\mathrm{ex}}(t,x,y)$ pointwise for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$, see Proposition 2.2.6, below. With this, the assumptions (2.1.9.Exist$_\varepsilon$), (2.1.12.Exist$_0$) and (2.1.14.Conv) of Main Theorem II are satisfied.

Proposition 2.2.6. *For $\mathbb{D}\in\mathrm{L}^\infty(\Omega;\mathrm{L}^\infty(\mathcal{Y}))$, we have $\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon\mathbb{D}^{\mathrm{ex}}(x,y)\to\mathbb{D}^{\mathrm{ex}}(x,y)$ pointwise for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$.*

Proof. Recalling the notations introduced in Subsection 1.2.1, we set $A_\varepsilon := \Omega_\varepsilon^-\times\mathcal{Y}$ and $B_\varepsilon := \mathbb{R}^d\backslash\Omega_\varepsilon^+\times\mathcal{Y}$ as well as $N_\varepsilon := \Omega_\varepsilon^+\backslash\Omega_\varepsilon^-\times\mathcal{Y}$. Then, it holds $A_\varepsilon\cap B_\varepsilon=\emptyset$ and $\overline{A_\varepsilon}\cup\overline{N_\varepsilon}\cup\overline{B_\varepsilon}=\mathbb{R}^d\times\mathcal{Y}$ as well as $\mathrm{vol}(N_\varepsilon)\to 0$ as $\varepsilon\to 0$.

Let an arbitrary point $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$ be given. Then there exists $\varepsilon_0>0$ such that $(x,y)\notin N_\varepsilon$ for all $\varepsilon\le\varepsilon_0$. Therefore it holds either $(x,y)\in A_\varepsilon$ or $(x,y)\in B_\varepsilon$. For $(x,y)\in A_\varepsilon$, the Lebesgue–Besicovitch differentiation theorem, cf. [EvG92, Thm. 1 p. 43], yields

$$(\mathcal{T}_\varepsilon D^\varepsilon)(x,y) = \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} \mathbb{D}(\xi,y)\,\mathrm{d}\xi \xrightarrow{\varepsilon\to 0} \mathbb{D}(x,y).$$

If $(x,y) \in B_\varepsilon$, then $(\mathcal{T}_\varepsilon D^\varepsilon)(x,y) = 0 = \mathbb{D}^{\mathrm{ex}}(x,y)$ and the proof is finished. $\square$

If the given data $\mathbb{D}_i, \mathbb{F}_i$ and V_0 are additionally continuous (either with respect to $x \in \Omega$ or $y \in \mathcal{Y}$), we can as well choose the "naive folding"

$$v_0^\varepsilon(x) := V_0(x, \tfrac{x}{\varepsilon}),\ D_i^\varepsilon(t,x) := \mathbb{D}_i(t,x,\tfrac{x}{\varepsilon}),\ \text{and } F_i^\varepsilon(t,x,A,B) := \mathbb{F}_i(t,x,\tfrac{x}{\varepsilon},A,B).$$

For a short discussion on admissible function spaces for two-scale convergence, we refer to [LNW02, Sec. 2].

In order to satisfy the additional assumptions (2.1.9.Time$_\varepsilon$) and (2.1.12.Time$_0$) in Main Theorem I, the choice of the initial values is more involved. For given right-hand side $f \in H$, we choose u_0^ε and u_0 as the unique (weak) solutions of the elliptic problems

$$u_0^\varepsilon - \operatorname{div}(D_1^\varepsilon(0)\nabla u_0^\varepsilon) = f \ \text{ in } \Omega \quad \text{and} \quad u_0 - \operatorname{div}(D_{\mathrm{eff}}(0)\nabla u_0) = f \ \text{ in } \Omega$$

supplemented with homogeneous Neumann boundary conditions on $\partial\Omega$. It is a well-known result in the theory of periodic homogenization that $u_0^\varepsilon \rightharpoonup u_0$ in X and $\nabla u_0^\varepsilon \xrightharpoonup{2\mathrm{w}} \nabla u_0 + \nabla_y U$ in $\mathbb{H}$. We have, in particular, the uniform boundedness of $\operatorname{div}(D_1^\varepsilon(0)\nabla u_0^\varepsilon)$ and $\operatorname{div}(D_{\mathrm{eff}}(0)\nabla u_0)$ in H.

We proceed analogously in the slow-diffusion case. For given $G \in \mathbb{H}$, we choose v_0^ε and V_0 as the unique (weak) solutions of

$$v_0^\varepsilon - \operatorname{div}(\varepsilon^2 D_2^\varepsilon(0)\nabla v_0^\varepsilon) = \mathcal{F}_\varepsilon G^{\mathrm{ex}} \ \text{ in } \Omega \quad \text{and} \quad V_0 - \operatorname{div}_y(\mathbb{D}_2\nabla_y V_0) = G \text{ in } \Omega\times\mathcal{Y}. \tag{2.2.21}$$

Proposition 2.2.7. *For v_0^ε and V_0 as in (2.2.21), we have $v_0^\varepsilon \xrightarrow{2\mathrm{s}} V_0$ in $\mathbb{X}$.*

Proof. It is well-known in the literature, cf. [All92, PeB08, Han11], that the sequence $(v_0^\varepsilon)_\varepsilon$ of solutions of $(2.2.21)_1$ are uniformly bounded in X_ε and that $v_0^\varepsilon \xrightharpoonup{2\mathrm{w}} V_0$ in $\mathbb{X}$, where V_0 solves $(2.2.21)_2$. The strong two-scale convergence follows from the estimate (cf. [Han11, Thm. 4.1 & Rem. 5.1])

$$\begin{aligned}
&\min\{\alpha,1\}\|\mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}}\|^2_{\mathrm{L}^2(\mathbb{R}^d;\mathrm{H}^1(Y))}\\
&\le \int_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}}) : \nabla_y(\mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}}) + (\mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}})^2 \,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\mathbb{R}^d\times\mathcal{Y}} \Big\{ \underbrace{\mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon v_0^\varepsilon) : \nabla_y(\mathcal{T}_\varepsilon v_0^\varepsilon) + (\mathcal{T}_\varepsilon v_0^\varepsilon)^2}_{=\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon G^{\mathrm{ex}}\cdot \mathcal{T}_\varepsilon v_0^\varepsilon} + \underbrace{\mathbb{D}_2^{\mathrm{ex}}\nabla_y V_0^{\mathrm{ex}} : \nabla_y V_0^{\mathrm{ex}} + (V_0^{\mathrm{ex}})^2}_{=G^{\mathrm{ex}}\cdot V_0^{\mathrm{ex}}}\\
&\qquad - \mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon v_0^\varepsilon) : \nabla_y V_0^{\mathrm{ex}} - \mathcal{T}_\varepsilon D_2^\varepsilon \nabla_y V_0^{\mathrm{ex}} : \nabla_y(\mathcal{T}_\varepsilon v_0^\varepsilon) - 2\,\mathcal{T}_\varepsilon v_0^\varepsilon \cdot V_0^{\mathrm{ex}}\\
&\qquad + (\mathcal{T}_\varepsilon D_2^\varepsilon - \mathbb{D}_2^{\mathrm{ex}})\nabla_y V_0^{\mathrm{ex}} : \nabla_y V_0^{\mathrm{ex}} \Big\} \,\mathrm{d}x\,\mathrm{d}y\\
&\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^d\times\mathcal{Y}} 2G^{\mathrm{ex}}\cdot V_0^{\mathrm{ex}} - 2\mathbb{D}_2^{\mathrm{ex}}\nabla_y V_0^{\mathrm{ex}} : \nabla_y V_0^{\mathrm{ex}} - 2(V_0^{\mathrm{ex}})^2 + 0 \,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned}$$

Exploiting strong/weak convergence for the duality pairing gives the limit. $\square$

2.3 Quantitative estimates

In this section, we prove convergence rates for the strong convergences derived in Section 2.1 and 2.2. Assuming additional regularity with respect to $x \in \Omega$ of the given data as well as of the solutions of the limit problem (2.0.2.P$_0$), we show

$$\| \mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}} \|_{C([0,T];\mathbb{H}_{\mathbb{R}^d})} + \|u^\varepsilon - u\|_{C([0,T];H)} \le \varepsilon^{1/4} C, \tag{2.3.1a}$$

$$\begin{aligned} &\| \mathcal{T}_\varepsilon(\varepsilon \nabla v^\varepsilon) - \nabla_y V^{\mathrm{ex}} \|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})} \\ &\quad + \| \mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\mathrm{ex}} \|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})} \le \varepsilon^{1/4} C. \end{aligned} \tag{2.3.1b}$$

We emphasize that we do not assume additional regularity with respect to the microscopic scale $y \in \mathcal{Y}$: neither for the given data, nor for the corrector functions. Based on estimate (2.0.5.Est), we proceed as in Subsection 2.1.6 on the control of the error terms Δ^{u^ε} and Δ^{v^ε}, and provide quantitative estimates. Throughout Section 2.3, we postulate the following assumption for the effective solution (u, V) and the given data. Let $i \in \{1, 2\}$.

The given data: $\mathbb{D}_i \in \mathcal{M}(\Omega \times \mathcal{Y})$ and $\mathbb{F}_i \in \mathcal{F}(\Omega \times \mathcal{Y})$. (2.3.2.A0)

Spatial Lipschitz continuity of the given data:
For all $j \in \{1, \dots, d\}$, it is $\partial_{x_j} \mathbb{D}_i \in C^1((0,T); \mathrm{L}^\infty(\Omega \times \mathcal{Y}))$
and $(t, x, y) \mapsto \partial_{x_j} \mathbb{F}_i(t, x, y, A, B) \in C^1((0,T); \mathrm{L}^\infty(\Omega \times \mathcal{Y}))$ for all $(A, B) \in \mathbb{R}^{m_1 + m_2}$. (2.3.2.A1)

The dependence on ε:
$D_i^\varepsilon(t, x) := \mathbb{D}_i(t, x, x/\varepsilon)$ and $F_i^\varepsilon(t, x, A, B) := \mathbb{F}_i(t, x, x/\varepsilon, A, B)$. (2.3.2.A2)

Improved regularity of the effective solutions:
$u \in \mathrm{L}^2(0,T; \mathrm{H}^2(\Omega)) \cap \mathrm{H}^1(0,T; \mathrm{L}^2(\Omega))$ and
$V \in \mathrm{L}^2(0,T; \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))) \cap \mathrm{H}^1(0,T; \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y})))$. (2.3.2.A3)

Choice of initial values:
For given $u_0 \in X$ and $V_0 \in \mathbb{X}$, there exists $c \ge 0$ such that
$\|u_0^\varepsilon\|_H + \| \operatorname{div}(D_1^\varepsilon(0) \nabla u_0^\varepsilon)\|_H + \|v_0^\varepsilon\|_H + \| \operatorname{div}(\varepsilon^2 D_2^\varepsilon(0) \nabla v^\varepsilon)\|_H \le c$ and
$\| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}} \|_{\mathbb{H}_{\mathbb{R}^d}} + \|u_0^\varepsilon - u_0\|_H \le \varepsilon^{1/4} c$. (2.3.2.A4)

The assumptions (2.3.2.A0), (2.3.2.A2), and (2.3.2.A4) immediately imply (2.1.9.Exist$_\varepsilon$) and (2.1.9.Time$_\varepsilon$) so that the existence of solutions to (2.0.1.P$_\varepsilon$) with $u^\varepsilon \in W_{\mathrm{imp}}(0,T;X)$ and $v^\varepsilon \in W_{\mathrm{imp}}(0,T;X_\varepsilon)$ is guaranteed. Equally, (2.3.2.A0) and (2.3.2.A4) yield the existence of a solution to (2.0.2.P$_0$) with $u \in W(0,T;X)$ and $V \in W(0,T;\mathbb{X})$. The assumption (2.3.2.A3) demands improved spatial and temporal regularity of (u, V), in particular, we seek $u_t(t) \in H$ and $V_t(t) \in \mathbb{H}$ for almost every $t \in [0,T]$. For the effective two-scale solution V the higher regularity with respect to $x \in \Omega$ follows from the additional regularity of the given data in (2.3.2.A1). The higher x-regularity of the one-scale solution u in (2.3.2.A3) is by no means trivial for general systems. However, if the boundary of Ω and the diffusion coefficients are smooth, then higher regularity results are possible, see Subsection 2.3.8. Note that (2.3.2.A0)–(2.3.2.A1) directly imply Lipschitz continuity for the macroscopic tensor, i.e. $D_{\mathrm{eff}} \in C^1([0,T]; W^{1,\infty}(\Omega))$.

Thanks to the better regularity of the given data in (2.3.2.A1), the choice of the "naive folding" $D^\varepsilon(x) = \mathbb{D}(x, x/\varepsilon)$ in (2.3.2.A2) (also called "macroscopic reconstruction" in [Eck05]) is well-defined. Overall, the assumptions of Section 2.1 to derive the main estimate (2.0.5.Est) as well as to control the error terms Δ^{u^ε} and Δ^{v^ε} are satisfied. In the following, we use the higher x-regularity of the data and the effective solutions in (2.3.2.A1) and (2.3.2.A3), respectively, to derive quantitative estimates for those error terms.

For the estimation of these terms, we distinguish the following three types of errors with $w^\varepsilon \in \{u^\varepsilon, v^\varepsilon\}$ and $\sigma, \eta \in \{0, 1\}$

$$\begin{aligned} &\text{approximation error for } i = 3, 4, 5: && |\Delta_i^{w^\varepsilon}| \le (\varepsilon + \sigma\sqrt{\varepsilon})C, \\ &\text{folding mismatch error:} && |\Delta_1^{w^\varepsilon}| \le (\varepsilon + \sigma\sqrt{\varepsilon})C, \\ &\text{periodicity defect error:} && |\Delta_2^{w^\varepsilon}| \le (\varepsilon + \eta\sqrt{\varepsilon})C. \end{aligned} \tag{2.3.3}$$

Here, $\sigma = 1$ accounts for the error of lower order at the boundary $\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-$ and it is $\sigma = 0$ if the domain Ω satisfies $\Omega_\varepsilon^- = \Omega_\varepsilon^+$. Due to the definition of the unfolding operator $\mathcal{T}_\varepsilon$, there occur no "overhanging supports" $\operatorname{supp}(\mathcal{T}_\varepsilon\, u) \cap \Omega \times \mathcal{Y} \neq \emptyset$ in the case $\sigma = 0$ and hence, better estimates are available. Whereas the estimations of the approximation errors are rather standard, the quantification of the folding mismatch is a new result.

We exploit the estimates for the periodicity defect in [Gri04, Gri05]. The Boolean value η accounts for zero boundary conditions on $\partial\Omega$, namely it is $\eta = 0$ if the corresponding effective solutions and its gradients belong to the space $\mathrm{H}_0^1(\Omega)$ and it is $\eta = 1$ otherwise. Therefore, σ and η describe two different properties of the system.

We can now state the two main results of Section 2.3. For the first result, we denote with $\Omega \subset \mathbb{R}^d$ as before a bounded domain with Lipschitz boundary and we consider solutions with improved time-regularity.

Theorem 2.3.1 (Main Theorem IIIa). *Let* $(u^\varepsilon, v^\varepsilon)$ *and* (u, V) *denote the solutions of* (2.0.1.P$_\varepsilon$) *and* (2.0.2.P$_0$)*, respectively, and let the assumptions* (2.3.2) *be satisfied. Then, there exists a constant* $C \ge 0$ *independent of* ε *such that* (2.3.1a)–(2.3.1b) *hold.*

It is to be expected that the convergence rate $\varepsilon^{1/4}$ can be improved to $\varepsilon^{1/2}$ for the classically diffusing species u^ε in the absence of slow diffusion. Moreover, if the domain satisfies $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ (and, hence, the folding mismatch is of order $O(\varepsilon)$), we may be able to prove the rate $\varepsilon^{1/2}$ for the slowly diffusing species v^ε, too. However, this condition somehow contradicts the assumption $u \in \mathrm{H}^2(\Omega)$, which usually requires smoother boundaries. Moreover, we prove in Theorem 2.3.13 the better rate $\varepsilon^{1/2}$ for v^ε, decoupled from u^ε, in the interior of the domain Ω. For a further discussion we refer to Subsection 2.3.9.

For the second main result, we weaken the assumption (2.3.2.A4) on the initial values $(u_0^\varepsilon, v_0^\varepsilon)$ so that (2.1.9.Time$_\varepsilon$) does not hold in general. So, we consider solutions $u^\varepsilon \in W(0, T; X)$ and $v^\varepsilon \in W(0, T; X_\varepsilon)$ without improved-time regularity and obtain the lower convergence rate $\varepsilon^{1/6}$. We follow the approach in Section 2.2 and we approximate general L^2-initial values with regularized initial values $(u_0^{\varepsilon,\delta}, v_0^{\varepsilon,\delta})$ so that the associated solutions satisfy $\|u_t^{\varepsilon,\delta}\|_H + \|v_t^{\varepsilon,\delta}\|_H \le C(\delta)$. These norms are in general unbounded as $\delta \to 0$ and they enter the estimation of the folding mismatch, namely $|\Delta_1^{w^{\varepsilon,\delta}}| \le (\varepsilon + \sigma\sqrt{\varepsilon})C(\delta)$. Choosing $\delta = \delta(\varepsilon)$ sufficiently small gives the following result.

Theorem 2.3.2 (Main Theorem IIIb)**.** *Let the assumptions* (2.3.2.A0)–(2.3.2.A3) *hold. Moreover, let the initial values satisfy:*

$$\begin{aligned}&\text{For given } u_0 \in X \text{ and } V_0 \in \mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y})), \text{ there exists } c \geq 0 \text{ such that}\\ &\|u_0^\varepsilon\|_X + \|v_0^\varepsilon\|_{X_\varepsilon} \leq c \text{ and } \|\mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u_0^\varepsilon - u_0\|_H \leq \varepsilon^{1/2}c.\end{aligned} \tag{2.3.4}$$

Then, the solutions $(u^\varepsilon, v^\varepsilon)$ *and* (u, V) *of* (2.0.1.P$_\varepsilon$) *and* (2.0.2.P$_0$)*, respectively, satisfy the estimates*

$$\begin{aligned}&\|\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}}\|_{C([0,T];\mathbb{H}_{\mathbb{R}^d})} + \|\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon) - \nabla_y V^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}\\ &\quad + \|u^\varepsilon - u\|_{C([0,T];H)} + \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})} \leq \varepsilon^{1/6}C.\end{aligned}$$

Note that the more general assumptions on the initial value (u_0, V_0) in (2.3.4) might not directly imply the improved regularity in (2.3.2.A3). However, the theory of maximal parabolic regularity might apply to special cases of the effective system (2.0.2.P$_0$). The resulting improved time-regularity $u_t(t) \in H$ can be used to derive $u(t) \in \mathrm{H}^2(\Omega)$ for $t \in (0, T]$, see Subsection 2.3.8.

Section 2.3 is structured as follows. We begin with the derivation of preparatory error estimates in order to prove quantitative estimates for the error terms Δ^{w^ε} in (2.3.3). More precisely, we study the approximation error in Subsection 2.3.1, the folding mismatch in Subsection 2.3.2, and the periodicity defect in Subsection 2.3.3. In the Subsections 2.3.4 and 2.3.5, we give the proof to the Main Theorems IIIa and IIIb, respectively. Then, we consider only slowly diffusing species in Subsection 2.3.6 and prove better convergence rates in the interior of the domain Ω. In Subsection 2.3.7 and 2.3.8, we comment on the choice of initial values as well as on the assumption on the improved regularity of the effective solutions. The section concludes with a discussion of related results in the literature in Subsection 2.3.9.

We use throughout Section 2.3 that $Y = [0,1)^d$ and $\mathrm{diam}(Y) = \sqrt{d}$.

2.3.1 Preparatory estimates I: approximation errors

In this subsection, we study the error which arises by approximating a given two-scale function with its folded one-scale counterpart respective macroscopic reconstruction. We recall that Ω is a bounded domain with Lipschitz boundary such that we have in general $\Omega_\varepsilon^- \subsetneqq \Omega \subsetneqq \Omega_\varepsilon^+$. The treatment of cells $\varepsilon(\lambda_i + Y)$ intersecting the boundary $\partial\Omega$ is crucial here and in the following subsections. Therefore, we begin with a rather classical result for the error on $\Omega \backslash \Omega_\varrho$, where $\Omega_\varrho = \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \varrho\}$, which is later on applied to the boundary cells $\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-$. This result is used several times in [Gri04, Gri05], too, and the following proof is based on personal communications with G. Griso.

Lemma 2.3.3. *For all* $u \in X$ *and* $U \in \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$*, there exists a constant* $C \geq 0$ *only depending on the properties of the domain* Ω *such that*

$$\|u\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varrho)} \leq \sqrt{\varrho}C\|u\|_{\mathrm{L}^2(\partial\Omega)} + \varrho C\|\nabla u\|_H$$

as well as

$$\|U\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varrho\times\mathcal{Y})} \leq \sqrt{\varrho}C\|U\|_{\mathrm{L}^2(\partial\Omega\times\mathcal{Y})} + \varrho C\|\nabla_x U\|_{\mathbb{H}}.$$

Proof. *Step 1.* Let $\varrho \in (0,1)$. We consider the open set $\mathcal{O} = (-1,1)^{d-1} \times (-1,1)$ in $\mathbb{R}^d$ and we define the subsets

$$\mathcal{O}_+ := (-1,1)^{d-1} \times (0,1), \quad \mathcal{O}_0 := (-1,1)^{d-1} \times \{0\} \ \text{ and } \ \mathcal{O}_\varrho := (-1,1)^{d-1} \times (0,\varrho).$$

Let us first consider $u \in C^\infty(\overline{\mathcal{O}_+})$. We have

$$\forall\, (x', x_d) \in \mathcal{O}_+ : \quad u(x', x_d) = u(x', 0) + \int_0^{x_d} \frac{\partial u}{\partial x_d}(x', t)\, \mathrm{d}t.$$

Using the substitution of variables $t = s x_d$ as well as $(a+b)^2 \le 2(a^2+b^2)$ gives

$$\forall\, (x', x_d) \in \mathcal{O}_\varrho : \quad |u(x', x_d)|^2 \le 2|u(x',0)|^2 + 2x_d \int_0^1 \left| \frac{\partial u}{\partial x_d}(x', s) \right|^2 \mathrm{d}s.$$

Then, integration with respect to x_d yields

$$\forall\, x' \in (-1,1)^{d-1} : \quad \int_0^\varrho |u(x', x_d)|^2 \,\mathrm{d}x_d \le 2\varrho |u(x',0)|^2 + \varrho^2 \int_0^1 \left| \frac{\partial u}{\partial x_d}(x', s) \right|^2 \mathrm{d}s.$$

Finally, we integrate with respect to x' so that

$$\|u\|^2_{\mathrm{L}^2(\mathcal{O}_\varrho)} \le 2\varrho \|u\|^2_{\mathrm{L}^2(\mathcal{O}_0)} + \varrho^2 \left\| \frac{\partial u}{\partial x_d} \right\|^2_{\mathrm{L}^2(\mathcal{O}_+)}. \tag{2.3.5}$$

Exploiting the dense embedding of $C^\infty(\overline{\mathcal{O}_+})$ into X, we have (2.3.5) for all $u \in X$.

Step 2. Let $u \in X$. There exists a finite covering $(\Omega_j)_j$ of the boundary $\partial\Omega$ such that for each j there exists a Lipschitz diffeomorphism θ_j which maps Ω_j to the open set $\mathcal{O}$ and $\Omega_j \cap \Omega$ to $\mathcal{O}_+$. To this covering of $\partial\Omega$, we associate a partition of unity

$$\phi_j \in C^1_c(\Omega_j; [0,1]) \quad \text{with} \quad \sum_j \phi_j(x) = 1 \quad \text{in a neighborhood of } \partial\Omega.$$

Then, the function $(\phi_j u) \circ \theta_j^{-1}$ belongs to $\mathrm{H}^1(\mathcal{O}_+)$ and satisfies (2.3.5) for ϱ small enough. Using the inverse diffeomorphism yields

$$\|u\|^2_{\mathrm{L}^2(\Omega \setminus \Omega_\varrho)} \le \varrho C \|u\|^2_{\mathrm{L}^2(\partial\Omega)} + \varrho^2 C \|\nabla u\|^2_H, \tag{2.3.6}$$

where $C \ge 0$ only depends on the properties of Ω.

Step 3. For $U \in \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y}))$, we have with (2.3.6)

$$\begin{aligned}
\|U\|^2_{\mathrm{L}^2(\Omega\setminus\Omega_\varrho \times \mathcal{Y})} &= \int_{\mathcal{Y}} \left(\int_{\Omega\setminus\Omega_\varrho} |U(x,y)|^2 \,\mathrm{d}x \right) \mathrm{d}y \\
&\le \int_{\mathcal{Y}} \left(\varrho C \|U(y)\|^2_{\mathrm{L}^2(\partial\Omega)} + \varrho^2 C \|\nabla_x U(y)\|^2_H \right) \mathrm{d}y \\
&\le \varrho C \|U\|^2_{\mathrm{L}^2(\partial\Omega \times \mathcal{Y})} + \varrho^2 C \|\nabla_x U\|^2_{\mathbb{H}},
\end{aligned}$$

which finishes the proof. □

Exploiting the continuous embeddings $X \subset \mathrm{L}^2(\partial\Omega)$ and $\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})) \subset \mathrm{L}^2(\partial\Omega \times \mathcal{Y})$, we obtain

$$\|u\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varrho)} \leq (\varrho + \sqrt{\varrho})C\|u\|_X \quad \text{and} \quad \|U\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varrho\times\mathcal{Y})} \leq (\varrho + \sqrt{\varrho})C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}.$$

The most important observation in deriving the error estimates (2.3.1a)–(2.3.1b) is the quantification of the well-known two-scale property, cf. Proposition 1.2.4(c), for every $U \in \mathrm{L}^2(\Omega \times \mathcal{Y})$ exists a sequence $(u^\varepsilon)_\varepsilon \subset \mathrm{L}^2(\Omega)$ such that $u^\varepsilon \xrightarrow{2\mathrm{s}} U$ in $\mathrm{L}^2(\Omega \times \mathcal{Y})$. For example, such a sequence is given by $u^\varepsilon = \mathcal{F}_\varepsilon U$. Based on the explicit definitions of $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$, the following lemma provides quantitative estimates.

Lemma 2.3.4. *For all $u \in X$ and $U \in \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$, there exists a constant $C \geq 0$ only depending on Ω and Y such that*

$$\begin{aligned}\|\mathcal{T}_\varepsilon u - u^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} &\leq (\varepsilon + \sigma\sqrt{\varepsilon})C\|u\|_X,\\ \|U^{\mathrm{ex}} - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} &\leq (\varepsilon + \sigma\sqrt{\varepsilon})C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))},\end{aligned}$$

where $\sigma = 0$ if $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ and $\sigma = 1$ otherwise.

Proof. We begin with the estimate for $U \in \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$. The Poincaré–Wirtinger inequality applied on each cell $\mathrm{int}(\lambda_i + \varepsilon Y) \subset \Omega_\varepsilon^-$ yields

$$\begin{aligned}\|U - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U\|^2_{\mathrm{L}^2(\Omega_\varepsilon^-\times\mathcal{Y})} &= \sum_{\lambda_i\in\Lambda_\varepsilon^-} \int_{\varepsilon(\lambda_i+Y)} \int_{\mathcal{Y}} \left| U(x,y) - \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} U(z,y)\,\mathrm{d}z \right|^2 \mathrm{d}x\,\mathrm{d}y\\ &\leq \sum_{\lambda_i\in\Lambda_\varepsilon^-} (\mathrm{diam}(\varepsilon(\lambda_i+Y))^2\, \|\nabla_x U\|^2_{\mathrm{L}^2(\varepsilon(\lambda_i+Y))} \leq \varepsilon^2 C\|\nabla_x U\|^2_{\mathbb{H}}.\end{aligned}$$

Here, we used that the Poincaré–Wirtinger constant is bounded by the diameter of the convex set $\varepsilon(\lambda_i + Y)$. Exploiting Lemma 2.3.3 with $\varrho = 2\varepsilon\sqrt{d}$ yields

$$\|U^{\mathrm{ex}} - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_{\mathrm{L}^2(\Omega_\varepsilon^+\setminus\Omega_\varepsilon^-\times\mathcal{Y})} \leq (\varepsilon + \sqrt{\varepsilon})C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))},$$

which gives the estimate for U. Now, let $u \in X$. Using the triangle inequality, the norm preservation of $\mathcal{T}_\varepsilon$ (cf. Proposition 1.2.1a), and repeating the previous arguments gives

$$\|\mathcal{T}_\varepsilon u - u^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} \leq \|u - \mathcal{F}_\varepsilon u^{\mathrm{ex}}\|_H + \|\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon u^{\mathrm{ex}} - u^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} \leq (\varepsilon + \sigma\sqrt{u})C\|u\|_X,$$

which finishes the proof. □

For $U \in C(\overline{\Omega};\mathrm{L}^2(\mathcal{Y}))$, we denote the naive folding or macroscopic reconstruction by

$$[U]^\varepsilon(x) := U(x, \tfrac{x}{\varepsilon}), \tag{2.3.7}$$

which is indeed well-defined, see e.g. [LNW02, Sect. 2]. For sufficiently smooth functions U, the two foldings $\mathcal{F}_\varepsilon U$ and $[U]^\varepsilon$ generate the same approximation error.

Lemma 2.3.5. *For all $U \in W^{1,\infty}(\overline{\Omega};\mathrm{L}^\infty(\mathcal{Y}))$, there exists a constant $C \geq 0$ only depending on Ω and Y such that*

$$\begin{aligned}\|U^{\mathrm{ex}} - \mathcal{T}_\varepsilon[U]^\varepsilon\|_{\mathrm{L}^\infty(\Omega_\varepsilon^-\times\mathcal{Y})} &\leq \varepsilon C\|\nabla_x U\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})},\\ \|U^{\mathrm{ex}} - \mathcal{T}_\varepsilon[U]^\varepsilon\|_{\mathbb{H}_{\mathbb{R}^d}} &\leq (\varepsilon + \sigma\sqrt{\varepsilon})C\|U\|_{W^{1,\infty}(\overline{\Omega};\mathrm{L}^\infty(\mathcal{Y}))},\end{aligned}$$

where $\sigma = 0$ if $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ and $\sigma = 1$ otherwise.

Proof. Exploiting the Lipschitz continuity of U with respect to $x \in \Omega$ gives

$$\sup_{(x,y)\in\Omega_\varepsilon^-\times\mathcal{Y}} |U(x,y) - U(\mathcal{N}_\varepsilon(x)+\varepsilon y, y)| \le \sup_{(x,y)\in\Omega_\varepsilon^-\times\mathcal{Y}} \varepsilon\sqrt{d}\sup_{z\in\Omega} |\nabla_x U(z,y)|$$

and hence the first estimate. Furthermore, the L^∞-boundedness yields with $\mathrm{vol}(\mathcal{Y}) = 1$

$$\|U\|^2_{\mathrm{L}^2(\Omega\backslash\Omega_\varepsilon^-\times\mathcal{Y})} \le \|U\|^2_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})}\mathrm{vol}(\Omega\backslash\Omega_\varepsilon^-).$$

Then, the second estimate follows with $\mathrm{vol}(\Omega\backslash\Omega_\varepsilon^-) \le \varepsilon 2\sqrt{d}\cdot \mathrm{meas}(\partial\Omega)$, where $\mathrm{meas}(\partial\Omega)$ denotes the $d-1$ dimensional surface measure of $\partial\Omega$. □

2.3.2 Preparatory estimates II: folding mismatch

In this section, we prove quantitative estimates for the *folding mismatch* $\Delta_1^{v^\varepsilon}$ respective $\Delta_1^{u^\varepsilon}$, e.g. the error between $\mathcal{F}_\varepsilon U$ and $\mathcal{G}_\varepsilon^1 U$ as well as $\mathcal{F}_\varepsilon(\nabla_y U)$ and $\varepsilon\nabla(\mathcal{G}_\varepsilon^1 U)$. This error is of order $\varepsilon + \sigma\sqrt{\varepsilon}$ with $\sigma = 0$ if $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ and $\sigma = 1$ otherwise.

For possibly discontinuous functions $U \in \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$, the naive folding $U(x, x/\varepsilon)$ may not be well-defined. However, for the quantification of the folding mismatch in Proposition 2.3.8, exactly such a folding will be employed. Therefore, we need a suitable regularization U_ε of U so that $\vartheta_\varepsilon(x) = U_\varepsilon(x, x/\varepsilon)$ is well-defined and the difference between $\mathcal{F}_\varepsilon U$ and ϑ_ε is of order ε. Therefore, we use in addition to $\mathcal{G}_\varepsilon^\gamma$ another regularization of the folding operator $\mathcal{F}_\varepsilon$, namely, the so-called *scale-splitting operator* $\mathcal{Q}_\varepsilon$, cf. [CDG02, CDG08, Gri04]. For $u \in \mathrm{L}^1(\Omega)$, the function $\mathcal{Q}_\varepsilon u$ is the $\mathcal{Q}_1$-Lagrangian interpolant of the discrete function $\mathcal{F}_\varepsilon u$. The precise definition of $\mathcal{Q}_\varepsilon$ (see Definition 2.3.6 below) is as in [CDG08, Def. 4.1].

For bounded domains Ω with Lipschitz boundary, we have in general $\Omega_\varepsilon^- \subsetneqq \Omega \subsetneqq \Omega_\varepsilon^+$. Since the folding operator $\mathcal{F}_\varepsilon$ and the gradient folding operator $\mathcal{G}_\varepsilon^\gamma$ are defined on Ω, only, we extend their definitions to Ω_ε^+. As in the previous section, we obtain an error of order ε on Ω_ε^+ and an additional boundary error term of order $\sqrt{\varepsilon}$ on $\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-$. By the Sobolev extension theorem, there exists a linear and continuous operator, cf. e.g. [Nečˇ67, Thm. 3.9],

$$\mathcal{P} : \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})) \to \mathrm{H}^1(\mathbb{R}^d;\mathrm{L}^2(\mathcal{Y}))$$

such that $\mathcal{P}(U)|_{\Omega\times\mathcal{Y}} = U$ and $\|\mathcal{P}(U)\|_{\mathrm{H}^1(\mathbb{R}^d;\mathrm{L}^2(\mathcal{Y}))} \le C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}$. Moreover, let $\mathcal{P}$ be such that its restriction to one-scale functions $u \in X$ satisfies $\mathcal{P}(u)|_\Omega = u$ and $\|\mathcal{P}(u)\|_{\mathrm{H}^1(\mathbb{R}^d)} \le C\|u\|_X$. Furthermore, we denote with

$$\mathcal{P}_2 : \mathrm{H}^2(\Omega) \to \mathrm{H}^2(\mathbb{R}^d)$$

the extension operator with $\mathcal{P}_2(u)|_\Omega = u$ and $\|\mathcal{P}_2(u)\|_{\mathrm{H}^2(\mathbb{R}^d)} \le C\|u\|_{\mathrm{H}^2(\Omega)}$. With this, we define the following folding operators.

Definition 2.3.6. *(a) We define* $\mathcal{F}_\varepsilon^+ : \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})) \to \mathrm{L}^\infty(\mathbb{R}^d)$ *via*

$$(\mathcal{F}_\varepsilon^+ U)(x) := \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} (\mathcal{P}U)\left(z, \{\tfrac{x}{\varepsilon}\}_Y\right)\,\mathrm{d}z.$$

(b) *The gradient folding operator* $\mathcal{G}_\varepsilon^{+,0} : \mathrm{H}^2(\Omega) \times \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})) \to \mathrm{H}^1(\Omega_\varepsilon^+)$ *maps a pair* $(u, U) \in \mathrm{H}^2(\Omega) \times \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ *to* $u^\varepsilon := \mathcal{G}_\varepsilon^1(u, U)$*, where* $u^\varepsilon \in \mathrm{H}^1(\Omega_\varepsilon^+)$ *is the unique solution of the elliptic problem*

$$\int_{\Omega_\varepsilon^+} (u^\varepsilon - \mathcal{P}_2(u)) \cdot \varphi + \Big(\nabla u^\varepsilon - \mathcal{F}_\varepsilon^+(\nabla u + \nabla_y U)\Big) : \nabla \varphi \, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathrm{H}^1(\Omega_\varepsilon^+).$$

Moreover, we define $\widetilde{\mathcal{G}}_\varepsilon^0(u, U) := u^\varepsilon|_\Omega$ *as the restriction of* $\mathcal{G}_\varepsilon^{+,0}(u, U)$ *on* Ω.

(c) *The gradient folding operator* $\mathcal{G}_\varepsilon^{+,1} : \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y})) \to \mathrm{H}^1(\Omega_\varepsilon^+)$ *maps a two-scale function* $U \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ *to* $u^\varepsilon := \mathcal{G}_\varepsilon^{+,1}(U)$*, where* $u^\varepsilon \in \mathrm{H}^1(\Omega_\varepsilon^+)$ *is the unique solution of the elliptic problem*

$$\int_{\Omega_\varepsilon^+} \Big(u^\varepsilon - \mathcal{F}_\varepsilon^+(U)\Big) \cdot \varphi + \Big(\varepsilon \nabla u^\varepsilon - \mathcal{F}_\varepsilon^+(\nabla_y U)\Big) : \varepsilon \nabla \varphi \, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathrm{H}^1(\Omega_\varepsilon^+).$$

Moreover, we define $\widetilde{\mathcal{G}}_\varepsilon^1(U) := u^\varepsilon|_\Omega$ *as the restriction of* $\mathcal{G}_\varepsilon^{+,1}(U)$ *on* Ω.

(d) *We define* $\mathcal{Q}_\varepsilon : X \to W^{1,\infty}(\mathbb{R}^d)$ *as follows: for* $x \in \mathcal{N}_\varepsilon(x) + \varepsilon Y$ *and every* $\kappa = (\kappa_1, \ldots, \kappa_d) \in \{0,1\}^d$*, we set*

$$\bar{x}_l^{(\kappa_l)} := \begin{cases} \frac{x_l - \mathcal{N}_\varepsilon(x)_l}{\varepsilon} & \textit{if } \kappa_l = 1 \\ 1 - \frac{x_l - \mathcal{N}_\varepsilon(x)_l}{\varepsilon} & \textit{if } \kappa_l = 0 \end{cases}$$

and

$$(\mathcal{Q}_\varepsilon \, w)(x) := \sum_{\kappa \in \{0,1\}^d} (\mathcal{F}_\varepsilon^+ w)(\mathcal{N}_\varepsilon(x) + \varepsilon\kappa) \cdot \bar{x}_1^{(\kappa_1)} \cdots \bar{x}_d^{(\kappa_d)}. \tag{2.3.8}$$

We briefly comment on the definitions of the new folding operators. By construction, we have $(\mathcal{F}_\varepsilon^+ U)|_{\Omega_\varepsilon^- \times \mathcal{Y}} = (\mathcal{F}_\varepsilon \, U)|_{\Omega_\varepsilon^- \times \mathcal{Y}}$ and $\|\mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq \|\mathcal{P} U\|_{\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})}$ by Jensen's inequality. The new gradient folding operator $\widetilde{\mathcal{G}}_\varepsilon^1$ maps two-scale functions $U \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ to one-scale functions $u^\varepsilon \in X_\varepsilon$. Since the effective solutions are by assumption of higher x-regularity, we can apply $\widetilde{\mathcal{G}}_\varepsilon^1$ instead of $\mathcal{G}_\varepsilon^1$ in the proof of (2.0.5.Est) and still obtain an admissible test function. In Proposition 2.3.8 below, we derive quantitative estimates for the differences $\mathcal{F}_\varepsilon \, U - \widetilde{\mathcal{G}}_\varepsilon^1 U$ and $\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla(\widetilde{\mathcal{G}}_\varepsilon^1 U)$ with the help of the previously introduced operator $\mathcal{Q}_\varepsilon$.

The interpolant $\mathcal{Q}_\varepsilon \, w$ is given on each node $x = \mathcal{N}_\varepsilon(x)$ via the average over the associated cell, namely, $(\mathcal{Q}_\varepsilon \, w)(x) = \fint_{\mathcal{N}_\varepsilon(x) + \varepsilon Y} w(z) \, \mathrm{d}z$. And on each cell $\mathcal{N}_\varepsilon(x) + \varepsilon Y$, $\mathcal{Q}_\varepsilon \, w$ interpolates the 2^d vertices of the cell via $\mathcal{Q}_1$-Lagrange elements as customary in the finite elements methods. The polynomial $\mathcal{Q}_\varepsilon \, w$ is of degree d on the interior of the cell and of degree $d{-}1$ along the edges of each cell, see e.g. [Bra07, ff. 64-66] or [GrR05, Sect. 4.2.2]. According to [CDG08, Prop. 4.5], there exists a constant $C \geq 0$ only depending on Ω, Y, and $\mathcal{P}$ such that

$$\| \mathcal{Q}_\varepsilon \, w\|_X \leq C \|w\|_X \quad \text{for all } w \in X. \tag{2.3.9}$$

Finally, we state one auxiliary lemma which we use to prove the error estimate for the folding mismatch afterward. Note that for general functions $w \in X$ and $z \in \mathrm{L}^2(\mathcal{Y})$, the compositions $x \mapsto (\mathcal{F}_\varepsilon^+ w)(x) z(x/\varepsilon)$ and $x \mapsto (\mathcal{Q}_\varepsilon \, w)(x) z(x/\varepsilon)$ belong to H, cf. [LNW02, Thm.4] and [Gri04, Prop. 3.2], respectively.

Lemma 2.3.7. *For $w \in X$ and $z \in \mathrm{L}^2(\mathcal{Y})$, there exists a constant $C \geq 0$ only depending on the dimension d and the operator $\mathcal{P}$ such that*

$$\|(\mathcal{F}_\varepsilon^+ w - \mathcal{Q}_\varepsilon\, w) z(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} \leq \varepsilon C \|w\|_X \|z\|_{\mathrm{L}^2(\mathcal{Y})}.$$

Proof. Based on the equality

$$\|(\mathcal{F}_\varepsilon^+ w - \mathcal{Q}_\varepsilon\, w) z(\tfrac{\cdot}{\varepsilon})\|^2_{\mathrm{L}^2(\Omega_\varepsilon^+)} = \sum_{\lambda_i \in \Lambda_\varepsilon^+} \int_{\varepsilon(\lambda_i + Y)} \left| \Big(\mathcal{F}_\varepsilon^+ w(x) - \mathcal{Q}_\varepsilon\, w(x) \Big)\, z(\tfrac{x}{\varepsilon}) \right|^2 \mathrm{d}x, \quad (2.3.10)$$

we consider in the following only one microscopic cell $\varepsilon(\lambda_i + Y)$ and without loss of generality we set $\lambda_i = 0$. Thus, we have by definition (2.3.8) for every $x \in \varepsilon Y$

$$\mathcal{F}_\varepsilon^+ w(x) - \mathcal{Q}_\varepsilon\, w(x) = \sum_{\kappa \in \{0,1\}^d} \Big[\mathcal{F}_\varepsilon^+ w(0) - \mathcal{F}_\varepsilon^+ w(\varepsilon\kappa) \Big]\, \bar{x}_1^{(\kappa_1)} \cdots \bar{x}_d^{(\kappa_d)}, \quad (2.3.11)$$

since $\sum_{\kappa \in \{0,1\}^d} \mathbb{1} \bar{x}_1^{(\kappa_1)} \cdots \bar{x}_d^{(\kappa_d)} = 1$ for the constant function $\mathbb{1}(x) \equiv 1$. Therefore, we continue to estimate the difference

$$\mathcal{F}_\varepsilon^+ w(0) - \mathcal{F}_\varepsilon^+ w(\varepsilon\kappa) = \fint_{\varepsilon Y} w(z) - w(z + \varepsilon\kappa)\, \mathrm{d}z. \quad (2.3.12)$$

(We recall that w is extended to $\mathbb{R}^d$ via the extension operator $\mathcal{P}$, but we suppress $\mathcal{P}$ in the notation here.) The fundamental theorem of calculus and substitution of variables give for every $z \in \varepsilon Y$

$$w(z + \varepsilon\kappa) - w(z) = \int_0^\varepsilon \nabla w(z + \tau\kappa) \cdot \kappa\, \mathrm{d}\tau = \varepsilon \int_0^1 \nabla w(z + \varepsilon t \kappa) \cdot \kappa\, \mathrm{d}t.$$

With this, (2.3.12) as well as $|\kappa| \leq \sqrt{d}$ and $\mathrm{vol}(\varepsilon Y) = \varepsilon^d$, we obtain

$$\begin{aligned} |\mathcal{F}_\varepsilon^+ w(0) - \mathcal{F}_\varepsilon^+ w(\varepsilon\kappa)|^2 &\leq \fint_{\varepsilon Y} \left(\varepsilon\sqrt{d} \int_0^1 |\nabla w(z + \varepsilon t\kappa)|\, \mathrm{d}t \right)^2 \mathrm{d}z \\ &\leq \frac{\varepsilon^2 d}{\varepsilon^d} \int_0^1 \left(\int_{\varepsilon Y} |\nabla w(z + \varepsilon t\kappa)|^2\, \mathrm{d}z \right) \mathrm{d}t = \frac{\varepsilon^2 d}{\varepsilon^d} \int_{\varepsilon Y} |\nabla w(\xi)|^2\, \mathrm{d}\xi. \end{aligned}$$

For the last equality, we used another substitution of variables with $|\det(\mathrm{d}\xi / \mathrm{d}z)| = 1$. With this and (2.3.11), we arrive at

$$\begin{aligned} &\int_{\varepsilon Y} \left| \Big(\mathcal{F}_\varepsilon^+ w(x) - \mathcal{Q}_\varepsilon\, w(x) \Big)\, z(\tfrac{x}{\varepsilon}) \right|^2 \mathrm{d}x \\ &= \int_{\varepsilon Y} \left(\sum_{\kappa \in \{0,1\}^d} \Big[\mathcal{F}_\varepsilon^+ w(0) - \mathcal{F}_\varepsilon^+ w(\varepsilon\kappa) \Big]\, \bar{x}_1^{(\kappa_1)} \cdots \bar{x}_d^{(\kappa_d)} \right)^2 |z(\tfrac{x}{\varepsilon})|^2\, \mathrm{d}x \\ &\leq 2^d \sum_{\kappa \in \{0,1\}^d} \Big[\mathcal{F}_\varepsilon^+ w(0) - \mathcal{F}_\varepsilon^+ w(\varepsilon\kappa) \Big]^2 \varepsilon^d \int_{\mathcal{Y}} |z(y)|^2\, \mathrm{d}y \\ &\leq 2^{2d} \varepsilon^2 d \|\nabla w\|^2_{\mathrm{L}^2(\varepsilon Y)} \|z\|^2_{\mathrm{L}^2(\mathcal{Y})}. \end{aligned}$$

Summing up over all $\lambda_i \in \Lambda_\varepsilon^+$ in (2.3.10) gives the desired estimate. □

The next proposition is a quantification of Proposition 1.2.9 (Comparison of $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^\gamma$, where $\gamma \in \{0,1\}$, cf. Definition 1.2.7 of $\mathcal{G}_\varepsilon^\gamma$) and the result seems to be unknown in periodic unfolding. It is applied to the estimation of the *folding mismatch* $\Delta_1^{u^\varepsilon}$ for $\gamma = 0$ respective $\Delta_1^{v^\varepsilon}$ for $\gamma = 1$.

Proposition 2.3.8. *For all* $(u, U) \in \mathrm{H}^2(\Omega) \times \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ *(*$\gamma = 0$*) respective* $U \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ *(*$\gamma = 1$*), there exists a constant* $C \geq 0$ *such that*

$$\begin{aligned}
\gamma = 0: \quad & \| \widetilde{\mathcal{G}}_\varepsilon^0(u, U) - u\|_H + \|\nabla[\widetilde{\mathcal{G}}_\varepsilon^0(u, U)] - \mathcal{F}_\varepsilon[\nabla u + \nabla_y U]^{\mathrm{ex}}\|_H \\
& \qquad\qquad \leq (\varepsilon + \sigma\sqrt{\varepsilon})C\big(\|u\|_{\mathrm{H}^2(\Omega)} + \|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))}\big); \\
\gamma = 1: \quad & \| \widetilde{\mathcal{G}}_\varepsilon^1 U - \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_H + \|\varepsilon\nabla[\widetilde{\mathcal{G}}_\varepsilon^1 U] - \mathcal{F}_\varepsilon[\nabla_y U]^{\mathrm{ex}}\|_H \leq (\varepsilon + \sigma\sqrt{\varepsilon})C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))},
\end{aligned}$$

where $\sigma = 0$ *if* $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ *and* $\sigma = 1$ *otherwise. The constant* C *depends on the domains* Ω*,* Y*, and the extension operators* $\mathcal{P}$*,* $\mathcal{P}_2$*.*

The proof contains the following three steps ($\gamma = 1$).

Step 1: Treatment of the boundary. Using the extended folding operators in Definition 2.3.6, we reduce the estimate from Ω to Ω_ε^+ and treat the boundary terms with Lemma 2.3.3. So, we do not care about intersected cells at the boundary anymore and prove the estimate

$$\|\mathcal{G}_\varepsilon^{+,1}U - \mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} + \|\varepsilon\nabla[\mathcal{G}_\varepsilon^{+,1}U] - \mathcal{F}_\varepsilon^+[\nabla_y U]\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} \leq \varepsilon C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))} \tag{2.3.13}$$

in the following two steps. Since $\mathcal{G}_\varepsilon^{+,1}U$, as the solution of the elliptic problem, has no higher regularity, we cannot cut off the boundary term on $\Omega\backslash\Omega_\varepsilon^-$. Nevertheless, using the extensions and the higher x-regularity of U, we can control the error term on $\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-$.

Step 2: Estimate (2.3.13) *for products* $U(x, y) = w(x)z(y)$*.* We split $u^\varepsilon := \mathcal{G}_\varepsilon^{+,1}U$ into a known part $\vartheta_\varepsilon(x) := (\mathcal{Q}_\varepsilon w)(x)z(x/\varepsilon)$ and a remainder part g_ε following [Han11, Prop. 2.1]. Using Lemma 2.3.7, we derive (2.3.13).
In the case of exact periodicity (i.e. $\mathbb{D}(x/\varepsilon)$) and classical diffusion, we know that the corrector U is of the form $U(x, y) = \sum_{j=1}^d \frac{\partial u}{\partial x_j}(x)z_j(y)$, whereas in the case of slow diffusion the effective solution $V(x, y)$ is in general not a product.

Step 3: Estimate (2.3.13) *for general functions* U*.* We exploit the tensor product structure of the space $\mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ and write general functions U in the form $U(x, y) = \sum_{j=1}^\infty u_j(x)\Phi_j(y)$, where $\{\Phi_j\}_j$ is an orthonormal basis in $\mathrm{H}^1(\mathcal{Y})$. We show the orthogonality $\mathcal{G}_\varepsilon^{+,1}\Phi_i \perp \mathcal{G}_\varepsilon^{+,1}\Phi_j$ in X_ε for $i \neq j$ and we use this as well as Parseval's identity to prove (2.3.13).

Proof of Proposition 2.3.8. The proof is adjusted to the case $\gamma = 1$ and it utilizes the gradient folding operator $\widetilde{\mathcal{G}}_\varepsilon^1$. In the case $\gamma = 0$, we resort to $\widetilde{\mathcal{G}}_\varepsilon^0$ and only point out the differences afterward.

Step 1: Treatment of the boundary. The triangle inequality and $(\mathcal{F}_\varepsilon^+ - \mathcal{F}_\varepsilon)|_{\Omega_\varepsilon^-} = \mathrm{id}$ give

$$\begin{aligned}
&\| \widetilde{\mathcal{G}}_\varepsilon^1 U - \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_H + \|\varepsilon\nabla[\widetilde{\mathcal{G}}_\varepsilon^1 U] - \mathcal{F}_\varepsilon[\nabla_y U]^{\mathrm{ex}}\|_H \\
&\leq \| \widetilde{\mathcal{G}}_\varepsilon^1 U - \mathcal{F}_\varepsilon^+ U\|_H + \|\varepsilon\nabla[\widetilde{\mathcal{G}}_\varepsilon^1 U] - \mathcal{F}_\varepsilon^+(\nabla_y U)\|_H \\
&\quad + \|\mathcal{F}_\varepsilon^+ U - \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_H + \|\mathcal{F}_\varepsilon^+(\nabla_y U) - \mathcal{F}_\varepsilon[\nabla_y U]^{\mathrm{ex}}\|_H \\
&\leq \|\mathcal{G}_\varepsilon^{+,1} U - \mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} + \|\varepsilon\nabla[\mathcal{G}_\varepsilon^{+,1} U] - \mathcal{F}_\varepsilon^+(\nabla_y U)\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} \\
&\quad + \| \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^-)} + \| \mathcal{F}_\varepsilon[\nabla_y U]^{\mathrm{ex}}\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^-)} \\
&\quad + \|\mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega_\varepsilon^+\setminus\Omega_\varepsilon^-)} + \|\mathcal{F}_\varepsilon^+(\nabla_y U)\|_{\mathrm{L}^2(\Omega_\varepsilon^+\setminus\Omega_\varepsilon^-)}.
\end{aligned}$$

Using Lemma 2.3.3 with $\varrho = \varepsilon\sqrt{d}$ respective $\varrho = 2\varepsilon\sqrt{d}$, we get

$$\| \mathcal{F}_\varepsilon U^{\mathrm{ex}}\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^-)} \leq \|U\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^-\times\mathcal{Y})} \leq \sqrt{\varepsilon}C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}$$

respective

$$\|\mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega_\varepsilon^+\setminus\Omega_\varepsilon^-)} \leq \|\mathcal{P}U\|_{\mathrm{L}^2(\Omega_\varepsilon^+\setminus\Omega_\varepsilon^-\times\mathcal{Y})} \leq \sqrt{\varepsilon}C\|\mathcal{P}U\|_{\mathrm{H}^1(\Omega_\varepsilon^+;\mathrm{L}^2(\mathcal{Y}))} \leq \sqrt{\varepsilon}C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}.$$

The analogous estimates hold for the gradient $\nabla_y U$. With this, we have derived the boundary estimate in the case $\sigma = 1$. In the following two steps, we show (2.3.13), viz.

$$\|\mathcal{G}_\varepsilon^{+,1} U - \mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} + \|\varepsilon\nabla[\mathcal{G}_\varepsilon^{+,1} U] - \mathcal{F}_\varepsilon^+[\nabla_y U]\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} \leq \varepsilon C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))}.$$

Step 2: Estimate (2.3.13) *for products $U(x,y) = w(x)z(y)$.* It remains to prove estimate (2.3.13) for domains which are the union of translated cells $\varepsilon(\lambda_i + Y)$. For notational simplicity, we write Ω, $\mathcal{F}_\varepsilon$, $\mathcal{G}_\varepsilon^1$ for Ω_ε^+, $\mathcal{F}_\varepsilon^+$, $\mathcal{G}_\varepsilon^{+,1}$ during the rest of the proof.

Let $U \in \mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ be a product of the form

$$U(x,y) = w(x)z(y) \quad \text{with} \quad w \in X \text{ and } z \in H^1(\mathcal{Y}). \tag{2.3.14}$$

Recalling the definitions of $\mathcal{G}_\varepsilon^1$ and $\mathcal{Q}_\varepsilon$ in (1.2.16) and (2.3.8), respectively, we decompose $u^\varepsilon := \mathcal{G}_\varepsilon^1 U \in X$ as follows

$$u^\varepsilon(x) = \vartheta_\varepsilon(x) + g_\varepsilon(x) \quad \text{with} \quad \vartheta_\varepsilon(x) = (\mathcal{Q}_\varepsilon w)(x)z(\tfrac{x}{\varepsilon}). \tag{2.3.15}$$

By construction, $\vartheta_\varepsilon \in X$ and we define $g_\varepsilon \in X$ as the solution of the elliptic problem

$$\begin{aligned}
&\int_\Omega g_\varepsilon \cdot \varphi + \varepsilon\nabla g_\varepsilon : \varepsilon\nabla\varphi \,\mathrm{d}x = \ell_\varepsilon(\varphi) \quad \text{for all } \varphi \in X, \text{ where} \qquad (2.3.16)\\
&\ell_\varepsilon(\varphi) = \int_\Omega (\mathcal{F}_\varepsilon U - \vartheta_\varepsilon) \cdot \varphi + (\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon) : \varepsilon\nabla\varphi \,\mathrm{d}x.
\end{aligned}$$

The function g_ε can be estimated as follows

$$\begin{aligned}
&\tfrac{1}{2}\left(\|g_\varepsilon\|_H + \|\varepsilon\nabla g_\varepsilon\|_H\right)^2 \leq \|g_\varepsilon\|_H^2 + \|\varepsilon\nabla g_\varepsilon\|_H^2 = \ell_\varepsilon(g_\varepsilon) \\
&\leq \left(\| \mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \| \mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon\|_H\right)\left(\|g_\varepsilon\|_H + \|\varepsilon\nabla g_\varepsilon\|_H\right), \qquad (2.3.17)
\end{aligned}$$

which yields $\|g_\varepsilon\|_H + \|\varepsilon\nabla g_\varepsilon\|_H \leq 2\,(\| \mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \| \mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon\|_H)$. Now, we estimate the difference of $u^\varepsilon = \mathcal{G}_\varepsilon^1 U$ and $\mathcal{F}_\varepsilon U$ by adding and subtracting ϑ_ε. Inserting

$\pm g_\varepsilon = \pm(u^\varepsilon - \vartheta_\varepsilon)$ and computing $\varepsilon\nabla\vartheta_\varepsilon(x) = \varepsilon\left(\nabla_x(\mathcal{Q}_\varepsilon w(x))\right) z(x/\varepsilon) + \mathcal{Q}_\varepsilon w(x)\nabla_y z(x/\varepsilon)$, we arrive at

$$\begin{aligned}
&\|u^\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon\nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H \\
&\le \|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|g_\varepsilon\|_H + \|\varepsilon\nabla\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \|\varepsilon\nabla g_\varepsilon\|_H \\
&\le 3\left(\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon\nabla\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H\right) \\
&= 3\Big(\|(\mathcal{Q}_\varepsilon w - \mathcal{F}_\varepsilon w)z(\cdot/\varepsilon)\|_H + \|(\mathcal{Q}_\varepsilon w - \mathcal{F}_\varepsilon w)\nabla_y z(\cdot/\varepsilon)\|_H \\
&\qquad + \varepsilon\left\|\left(\nabla_x(\mathcal{Q}_\varepsilon w(x))\right) z(\cdot/\varepsilon)\right\|_H\Big) \\
&\le \varepsilon C\left(\|w\|_X\|z\|_{\mathrm{H}^1(\mathcal{Y})} + \|w\|_X\right).
\end{aligned} \tag{2.3.18}$$

In the last inequality, we used Lemma 2.3.7 (where $\Omega = \Omega_\varepsilon^+$) and the boundedness (2.3.9) of $\mathcal{Q}_\varepsilon$. Thus, estimate (2.3.13) is proved for functions U of product form.

Step 3: Estimate (2.3.13) *for general functions U.* We justify the decomposition in (2.3.14). Let $\{\Phi_i\}_{i=1}^\infty$ be an orthonormal basis in $\mathrm{H}^1(\mathcal{Y})$ which is also orthogonal in $\mathrm{L}^2(\mathcal{Y})$. Then, we can express $U \in \mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ via the linear combination

$$U(x,y) = \sum_{i=1}^\infty u_i(x)\Phi_i(y), \quad \text{where } u_i(x) := \fint_{\mathcal{Y}} U(x,y)\Phi_i(y)\,\mathrm{d}y. \tag{2.3.19}$$

By construction, we have $u_i \in X$ as well as Parseval's identity

$$\|U\|^2_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))} = \sum_{i=1}^\infty \|u_i\|_X^2.$$

Moreover, we set $U_i(x,y) := u_i(x)\Phi_i(y)$.

By the assumptions on Φ_i, we have $\Phi_i(\cdot/\varepsilon) \perp \Phi_j(\cdot/\varepsilon)$ and $\nabla_y\Phi_i(\cdot/\varepsilon) \perp \nabla_y\Phi_j(\cdot/\varepsilon)$ in H for all $i \neq j$. Using $\Omega = \Omega_\varepsilon^+$ again, a substitution of variables yields

$$\begin{aligned}
\int_\Omega \Phi_i(\tfrac{x}{\varepsilon})\cdot\Phi_j(\tfrac{x}{\varepsilon})\,\mathrm{d}x &= \sum_{\lambda_i\in\Lambda_\varepsilon^+}\int_{\varepsilon(\lambda_i+Y)} \Phi_i(\tfrac{x}{\varepsilon})\cdot\Phi_j(\tfrac{x}{\varepsilon})\,\mathrm{d}x \\
&= \sum_{\lambda_i\in\Lambda_\varepsilon^+}\frac{1}{\varepsilon^d}\int_{\lambda_i+Y}\Phi_i(y)\cdot\Phi_j(y)\,\mathrm{d}y = 0
\end{aligned} \tag{2.3.20}$$

and analogously for $\nabla_y\Phi_i$. In particular, we have that $\{\Phi_i(\cdot/\varepsilon)\}_i$ is a orthogonal system in X for each $\varepsilon > 0$. Applying the folding operator $\mathcal{F}_\varepsilon$ to U_i gives

$$(\mathcal{F}_\varepsilon U_i)(x) = (\mathcal{F}_\varepsilon u_i)(x)\Phi_i(\tfrac{x}{\varepsilon}) \quad \text{and} \quad (\mathcal{F}_\varepsilon[\nabla_y U_i])(x) = (\mathcal{F}_\varepsilon u_i)(x)\nabla_y\Phi_i(\tfrac{x}{\varepsilon}).$$

Since $(\mathcal{F}_\varepsilon u_i)(x) = \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} u_i(z)\,\mathrm{d}z$ is constant in each cell $\varepsilon(\lambda_i + Y)$, we have as well $\mathcal{F}_\varepsilon U_i \perp \mathcal{F}_\varepsilon U_j$ and $\mathcal{F}_\varepsilon(\nabla_y U_i) \perp \mathcal{F}_\varepsilon(\nabla_y U_j)$ in H for all $i \neq j$. Therefore, it suffices to consider the basis functions Φ_i. By the definition of $\mathcal{G}_\varepsilon^1$, we have for $v_i^\varepsilon := \mathcal{G}_\varepsilon^1\Phi_i$

$$\int_\Omega \left(v_i^\varepsilon - \Phi_i(\tfrac{\cdot}{\varepsilon})\right)\cdot\varphi + \left(\varepsilon\nabla v_i^\varepsilon - \nabla_y\Phi_i(\tfrac{\cdot}{\varepsilon})\right) : \varepsilon\nabla\varphi\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in X_\varepsilon. \tag{2.3.21}$$

Choosing the test function $\varphi(x) = \Phi_j(x/\varepsilon)$ in (2.3.21) yields with (2.3.20) for all $i \neq j$

$$\int_\Omega v_i^\varepsilon\cdot\Phi_j(\tfrac{\cdot}{\varepsilon}) + \varepsilon\nabla v_i^\varepsilon : \nabla_y\Phi_j(\tfrac{\cdot}{\varepsilon})\,\mathrm{d}x = 0.$$

Furthermore, choosing the test function $\varphi(x) = v_j^\varepsilon(x)$ in (2.3.21) yields with the previous estimate

$$\int_\Omega v_i^\varepsilon \cdot v_j^\varepsilon + \varepsilon\nabla v_i^\varepsilon : \varepsilon\nabla v_j^\varepsilon \,\mathrm{d}x = 0.$$

With this, we conclude that $v_i^\varepsilon \perp v_j^\varepsilon$ in X_ε for all $i \neq j$. We continue to estimate the folding mismatch of $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^1$ and set $u_i^\varepsilon := \mathcal{G}_\varepsilon^1 U_i$. Replacing $\mathcal{F}_\varepsilon \Phi_i$ and $\mathcal{F}_\varepsilon(\nabla_y \Phi_i)$ with $\mathcal{F}_\varepsilon U_i$ and $\mathcal{F}_\varepsilon(\nabla_y U_i)$ in (2.3.21), respectively, and exploiting that $\mathcal{F}_\varepsilon u_i$ is constant on each cell $\varepsilon(\lambda_i + Y)$ implies $u_i^\varepsilon \perp u_j^\varepsilon$ in X_ε for all $i \neq j$. Finally, we apply the result of Step 2 with $w = u_i$ and $z = \Phi_i$.

Since $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^1$ are linear and continuous operators, we have in particular

$$\mathcal{G}_\varepsilon^1 \left(\textstyle\sum_{i=1}^\infty U_i\right) = \textstyle\sum_{i=1}^\infty \mathcal{G}_\varepsilon^1 U_i \quad \text{and, hence,} \quad u^\varepsilon = \textstyle\sum_{i=1}^\infty u_i^\varepsilon.$$

Therefore, we square estimate (2.3.18) so that the mixed product terms vanish for $i \neq j$, namely

$$\begin{aligned}
&\|u^\varepsilon - \mathcal{F}_\varepsilon U\|_H^2 + \|\varepsilon\nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H^2 \\
&= \|\textstyle\sum_{i=1}^\infty (u_i^\varepsilon - \mathcal{F}_\varepsilon U_i)\|_H^2 + \|\textstyle\sum_{i=1}^\infty (\varepsilon\nabla u_i^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U_i))\|_H^2 \\
&= \textstyle\sum_{i=1}^\infty \|u_i^\varepsilon - \mathcal{F}_\varepsilon U_i\|_H^2 + \textstyle\sum_{i=1}^\infty \|\varepsilon\nabla u_i^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U_i)\|_H^2 \\
&\leq \textstyle\sum_{i=1}^\infty \varepsilon^2 C\|u_i\|_X^2 \|\Phi_i\|_{\mathrm{H}^1(\mathcal{Y})}^2 = \varepsilon^2 C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))}^2.
\end{aligned}$$

The last equality follows by Parseval's identity and thus, estimate (2.3.13) is proved for general functions U.

The case $\gamma = 0$: Step 1 holds analogously and, thus, $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ can be assumed without loss of generality. Note that if $u^\varepsilon \in X$ solves the elliptic problem

$$\int_\Omega (u^\varepsilon - u - \varepsilon\,\mathcal{F}_\varepsilon U)\cdot\varphi + (\nabla u^\varepsilon - [\nabla u + \mathcal{F}_\varepsilon(\nabla_y U)]) : \nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in X, \tag{2.3.22}$$

it holds by Proposition 1.2.1(b) and the Poincaré–Wirtinger inequality, cf. Lemma 2.3.4,

$$\|u^\varepsilon - \mathcal{G}_\varepsilon^0(u,U)\|_X \leq \|\varepsilon\,\mathcal{F}_\varepsilon U\|_H + \|\nabla u - \mathcal{F}_\varepsilon(\nabla u)\|_H \leq \varepsilon C(\|u\|_{\mathrm{H}^2(\Omega)} + \|U\|_{\mathbb{H}}).$$

With this, we can redefine $\mathcal{G}_\varepsilon^0(u,U) =: u^\varepsilon$ as the solution of (2.3.22). The linearity of the gradient folding operator and (2.3.22) imply $\mathcal{G}_\varepsilon^0(u,U) = \mathcal{G}_\varepsilon^0(u,0) + \mathcal{G}_\varepsilon^0(0,U)$ and $\mathcal{G}_\varepsilon^0(u,0) = u$, respectively. Hence, the estimate for $\gamma = 0$ reduces to

$$\|\,\mathcal{G}_\varepsilon^0(0,U) - \varepsilon\,\mathcal{F}_\varepsilon U\|_H + \|\nabla\,\mathcal{G}_\varepsilon^0(0,U) - \mathcal{F}_\varepsilon(\nabla_y U)\|_H \leq \varepsilon C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))}. \tag{2.3.23}$$

As in Step 2, let U be the product $U(x,y) = w(x)z(y)$ and the modifications of the proof are as follows. In (2.3.15), we set $u^\varepsilon := \mathcal{G}_\varepsilon^0(0,U)$ and decompose $u^\varepsilon = \varepsilon\vartheta_\varepsilon + g_\varepsilon$, where $\vartheta_\varepsilon(x) = (\mathcal{Q}_\varepsilon\, w)(x)z(x/\varepsilon)$ with $w \in X$ and $z \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$.

In (2.3.16), we use $(g_\varepsilon, \varphi)_X = \ell_\varepsilon(\varphi)$ for all $\varphi \in X$ with

$$\ell_\varepsilon(\varphi) = \int_\Omega (\varepsilon\,\mathcal{F}_\varepsilon U - \varepsilon\vartheta_\varepsilon)\cdot\varphi + (\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon) : \nabla\varphi \,\mathrm{d}x.$$

As in (2.3.17), it holds $\|g_\varepsilon\|_X \le 2\,(\varepsilon\|\,\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \|\,\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon\|_H)$. With this, we have analogously to (2.3.18),

$$\begin{aligned}&\|u^\varepsilon - \varepsilon\,\mathcal{F}_\varepsilon U\|_H + \|\nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H\\ &\le 3(\varepsilon\|\vartheta_\varepsilon\|_H + \varepsilon\|U\|_{\mathbb{H}} + \|\nabla_y\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \varepsilon\|\nabla_x\vartheta_\varepsilon\|_H).\end{aligned}$$

Again, the application of Lemma 2.3.7 and the boundedness (2.3.9) with the improved regularity $U \in \mathrm{H}^1(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ give estimate (2.3.23) for products U.

In Step 3, we decompose $U(x,y)$ as in (2.3.19) with $\Phi_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$. As in (2.3.21), it holds by the redefinition (2.3.22) for $v_i^\varepsilon := \mathcal{G}^0_\varepsilon(0,\Phi_i)$

$$\int_\Omega (v_i^\varepsilon - \varepsilon\Phi_i(\tfrac{\cdot}{\varepsilon}))\cdot\varphi + \big(\nabla v_i^\varepsilon - \nabla_y\Phi_i(\tfrac{\cdot}{\varepsilon})\big) : \nabla\varphi\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in X.$$

Following further the argumentation of Step 3 by inserting first $\varphi(x) = \varepsilon\Phi_j(x/\varepsilon)$ and second $\varphi(x) = v_j^\varepsilon$ yields the desired orthogonalities, in particular, it holds $v_i^\varepsilon \perp v_j^\varepsilon$ in X for all $i \ne j$. With this, we obtain estimate (2.3.23) for general functions $U(x,y)$. □

2.3.3 Preparatory estimates III: periodicity defect

We use the following two lemmata from [Gri04, Gri05] for the estimation of the *periodicity defect* $\Delta_2^{u^\varepsilon}$ respective $\Delta_2^{v^\varepsilon}$. The first estimates involve the $X^* = (\mathrm{H}^1(\Omega))^*$-norm and are of order $\sqrt{\varepsilon}$, whereas the second estimates use the $\mathrm{H}^{-1}(\Omega) = (\mathrm{H}^1_0(\Omega))^*$-norm and yield the better error ε. Here, we use the standard notation $\mathrm{H}^1_0(\Omega) := \{u \in \mathrm{H}^1(\Omega) \,|\, u = 0 \text{ a.e. on } \partial\Omega\}$. Therefore, the second estimates (Lemma 2.3.10) are significantly more restrictive in their application, since they require that the solutions of (2.0.1.P_ε) satisfy the correct boundary conditions such as $\nabla u \in \mathrm{H}^1_0(\Omega)$ and $V \in \mathrm{H}^1_0(\Omega;\mathrm{H}^1(\mathcal{Y}))$. We recall that $\mathrm{L}^2(Y)$ and $\mathrm{L}^2(\mathcal{Y})$ can be identified, while $\mathrm{H}^1(\mathcal{Y})$ is a closed subspace of $\mathrm{H}^1(Y)$.

We emphasize that the error of lower order $\sqrt{\varepsilon}$ is not due to the discrepancy $\Omega_\varepsilon^- \ne \Omega_\varepsilon^+$ as in the subsections before. Assuming $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ does not automatically improve the order of the error from $\sqrt{\varepsilon}$ to ε, but assuming zero boundary conditions on $\partial\Omega$ indeed yields the better order ε.

Lemma 2.3.9. *For every $u \in X$ with $\|u\|_X \le c$ ($\gamma = 0$) respective $\|u\|_{X_\varepsilon} \le c$ ($\gamma = 1$), there exists a function $\Psi^\varepsilon \in \mathbb{X}_0$ respective $\Psi^\varepsilon \in \mathbb{X}$ and a constant $C \ge 0$ only depending on Ω and Y such that*

$$\begin{aligned}\gamma = 0:&\quad \|\Psi^\varepsilon\|_{\mathbb{X}} \le C\|u\|_X \text{ and } \|\,\mathcal{T}_\varepsilon(\nabla u) - [\nabla u + \nabla_y\Psi^\varepsilon]\|_{\mathrm{L}^2(\mathcal{Y};X^*)} \le (\varepsilon + \sqrt{\varepsilon})C\|u\|_X,\\ \gamma = 1:&\quad \|\Psi^\varepsilon\|_{\mathbb{X}} \le C\|u\|_{X_\varepsilon} \text{ and } \|\,\mathcal{T}_\varepsilon u - \Psi^\varepsilon\|_{\mathrm{H}^1(Y;X^*)} \le (\varepsilon + \sqrt{\varepsilon})C\|u\|_{X_\varepsilon}.\end{aligned}$$

Proof. Following [Gri04, Gri05], we define the sets

$$\widetilde{\Omega}_\varepsilon := \{x \in \mathbb{R}^d \,|\, \mathrm{dist}(x,\Omega) < 3\sqrt{d}\varepsilon\} \quad \text{and} \quad \bar{\Omega}_\varepsilon := \{x \in \mathbb{R}^d \,|\, \mathrm{dist}(x,\partial\Omega) < 3\sqrt{d}\varepsilon\}.$$

According to [Gri04, Sect. 3] or [Gri05, Lem. 2.1], there exists a linear and continuous extension operator $\mathcal{P}_\varepsilon$ from $\mathrm{H}^1(\Omega)$ to $\mathrm{H}^1(\widetilde{\Omega}_\varepsilon)$ such that

$$\begin{aligned}&\mathcal{P}_\varepsilon(u)_{|\Omega} = u, \quad \|\nabla\mathcal{P}_\varepsilon(u)\|_{\mathrm{L}^2(\widetilde{\Omega}_\varepsilon)} \le C\|\nabla u\|_H,\\ &\|\mathcal{P}_\varepsilon(u)\|_{\mathrm{L}^2(\widetilde{\Omega}_\varepsilon)} \le C\{\|u\|_H + \varepsilon\|\nabla u\|_{\mathrm{L}^2(\Omega\setminus\bar{\Omega}_\varepsilon)}\},\end{aligned} \tag{2.3.24}$$

where C only depends on d and $\partial\Omega$. Now, let $u \in X_\varepsilon$ with $\|u\|_{X_\varepsilon} \leq c$ ($\gamma = 1$). Then, [Gri05, Thm. 2.2] yields the existence of a two-scale function $\Psi^\varepsilon \in \mathbb{X}$ such that $\|\Psi^\varepsilon\|_{\mathbb{X}} \leq C\|u\|_{X_\varepsilon}$ and

$$\begin{aligned} \| \mathcal{T}_\varepsilon u - \Psi^\varepsilon\|_{\mathrm{H}^1(Y;X^*)} &\leq \varepsilon C\|u\|_{X_\varepsilon} + \sqrt{\varepsilon}C \left\{\|\mathcal{P}_\varepsilon(u)\|_{\mathrm{L}^2(\bar\Omega_\varepsilon)} + \varepsilon\|\nabla\mathcal{P}_\varepsilon(u)\|_{\mathrm{L}^2(\bar\Omega_\varepsilon)}\right\} \\ &\leq \varepsilon C\|u\|_{X_\varepsilon} + \sqrt{\varepsilon}C \left\{\|u\|_{X_\varepsilon} + \varepsilon\|\nabla u\|_H\right\}, \end{aligned}$$

where we used $\bar\Omega_\varepsilon \subset \widetilde\Omega_\varepsilon$ as well as estimate (2.3.24) and C only depends on d and $\partial\Omega$. Hence, the estimate for $\gamma = 1$ is shown.

The estimate for $\gamma = 0$ follows analogously from [Gri05, Thm. 2.3], i.e. for any $u \in X$, there exists a two-scale function $\Psi^\varepsilon \in \mathbb{X}$ such that $\|\Psi^\varepsilon\|_{\mathbb{X}} \leq C\|u\|_X$ and

$$\| \mathcal{T}_\varepsilon(\nabla u) - [\nabla u + \nabla_y\Psi^\varepsilon]\|_{\mathrm{L}^2(\mathcal{Y};X^*)} \leq C\left\{\varepsilon\|\nabla u\|_H + \sqrt{\varepsilon}\|\nabla\mathcal{P}_\varepsilon(u)\|_{\mathrm{L}^2(\bar\Omega_\varepsilon)}\right\}.$$

Since the latter estimate only involves the gradient $\nabla_y\Psi^\varepsilon$, we can find $\Psi^\varepsilon \in \mathbb{X}_0$, which finishes the proof. $\square$

Lemma 2.3.10 ([Gri04, Prop. 3.3 and Thm. 3.4])**.** *For every $u \in X$ with $\|u\|_X \leq c$ ($\gamma = 0$) respective $\|u\|_{X_\varepsilon} \leq c$ ($\gamma = 1$), there exists a function $\Psi^\varepsilon \in \mathbb{X}_0$ respective $\Psi^\varepsilon \in \mathbb{X}$ and a constant $C \geq 0$ only depending on Ω and Y such that*

$$\begin{aligned} \gamma = 0: &\quad \|\Psi^\varepsilon\|_{\mathbb{X}} \leq C\|u\|_X \text{ and } \| \mathcal{T}_\varepsilon(\nabla u) - [\nabla u + \nabla_y\Psi^\varepsilon]\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^{-1}(\Omega))} \leq \varepsilon C\|u\|_X, \\ \gamma = 1: &\quad \|\Psi^\varepsilon\|_{\mathbb{X}} \leq C\|u\|_{X_\varepsilon} \text{ and } \| \mathcal{T}_\varepsilon u - \Psi^\varepsilon\|_{\mathrm{H}^1(Y;\mathrm{H}^{-1}(\Omega))} \leq \varepsilon C\|u\|_{X_\varepsilon}. \end{aligned}$$

Remark 2.3.11. *We point out that the periodic unfolding operator in [Gri04, Gri05] is defined slightly different close to the boundary $\partial\Omega$, namely $\mathcal{T}_\varepsilon^{\mathrm{G}} : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega\times\mathcal{Y})$ with $(\mathcal{T}_\varepsilon^{\mathrm{G}} u)(x,y) = u(\mathcal{N}_\varepsilon(x){+}\varepsilon y)$ if $(x,y) \in \Omega_\varepsilon^-{\times}\mathcal{Y}$ and $(\mathcal{T}_\varepsilon^{\mathrm{G}} u)(x,y) = 0$ if $(x,y) \in \Omega\backslash\Omega_\varepsilon^-{\times}\mathcal{Y}$. Whereas in this text, we use the definition from [MiT07] and we recall $\mathcal{T}_\varepsilon^{\mathrm{M}} : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$ with $(\mathcal{T}_\varepsilon^{\mathrm{M}} u)(x,y) = u^{\mathrm{ex}}(\mathcal{N}_\varepsilon(x) + \varepsilon Y)$ so that $\mathrm{supp}(\mathcal{T}_\varepsilon^{\mathrm{M}} u) \subset \overline{\Omega_\varepsilon^+}\times\mathcal{Y}$. In particular, we have $\mathcal{T}_\varepsilon^{\mathrm{M}} u = \mathcal{T}_\varepsilon^{\mathrm{G}} u$ almost everywhere in $\Omega_\varepsilon^-\times\mathcal{Y}$.*

Nevertheless, this discrepancy does not change or weaken the error estimates in Lemma 2.3.9 and 2.3.10, above. Indeed, we have with Lemma 2.3.3 for every $u \in H$

$$\begin{aligned} \| \mathcal{T}_\varepsilon^{\mathrm{M}} u - \mathcal{T}_\varepsilon^{\mathrm{G}} u\|_{\mathrm{L}^2(\mathcal{Y};X^*)} &= \sup_{\|\Phi\|_{\mathbb{X}}=1} \left|\int_{(\Omega\backslash\Omega_\varepsilon^-)\times\mathcal{Y}} (\mathcal{T}_\varepsilon^{\mathrm{M}} u - \mathcal{T}_\varepsilon^{\mathrm{G}} u)\cdot\Phi\,\mathrm{d}x\,\mathrm{d}y\right| \\ \leq \sup_{\|\Phi\|_{\mathbb{X}}=1} \| \mathcal{T}_\varepsilon^{\mathrm{M}} u &- \mathcal{T}_\varepsilon^{\mathrm{G}} u\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varepsilon^-\times\mathcal{Y})}\|\Phi\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varepsilon^-\times\mathcal{Y})} \leq (\varepsilon+\sqrt{\varepsilon})C\|u\|_H. \end{aligned}$$

In the same manner, we obtain with Lemma 2.3.3 for $\Phi = 0$ a.e. on $\partial\Omega\times\mathcal{Y}$

$$\| \mathcal{T}_\varepsilon^{\mathrm{M}} u - \mathcal{T}_\varepsilon^{\mathrm{G}} u\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^{-1}(\Omega))} \leq \varepsilon C\|u\|_H.$$

Remark 2.3.12. *We briefly comment on how the space $\mathrm{H}^1(Y;X^*)$ in Lemma 2.3.9 (and analogously $\mathrm{H}^1(Y;\mathrm{H}^{-1}(\Omega))$ in Lemma 2.3.10) is connected to previously mentioned two-scale spaces. Thanks to the underlying tensor product structure of the two separable Hilbert spaces $X = \mathrm{H}^1(\Omega)$ and $\mathrm{L}^2(\mathcal{Y}) \cong \mathrm{L}^2(Y) \cong (\mathrm{L}^2(Y))^*$, it holds $\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})) \cong \mathrm{L}^2(Y;X)$ as* *well as* $\big(\mathrm{L}^2(\mathcal{Y};X)\big)^* \cong \mathrm{L}^2(Y;X^*)$. *Later on, we will estimate the periodicity defect for the slowly diffusing variable* v^ε *as follows: for arbitrary* $F,G\in \mathrm{L}^2(Y;X)$, *it is*

$$\begin{aligned}&\left|\int_{\Omega\times\mathcal{Y}} F\cdot(\mathcal{T}_\varepsilon v^\varepsilon-\Psi^\varepsilon)+G:\nabla_y(\mathcal{T}_\varepsilon v^\varepsilon-\Psi^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\right|\\&\le \|F\|_{\mathrm{L}^2(Y;X)}\|\mathcal{T}_\varepsilon v^\varepsilon-\Psi^\varepsilon\|_{\mathrm{L}^2(Y;X^*)}+\|G\|_{\mathrm{L}^2(Y;X)}\|\nabla_y(\mathcal{T}_\varepsilon v^\varepsilon-\Psi^\varepsilon)\|_{\mathrm{L}^2(Y;X^*)}.\end{aligned}$$

2.3.4 Proof of Main Theorem IIIa

The proof of Theorem 2.3.1 is structured as follows.

Step 1+2: Quantification of the error terms. Based on estimate (2.0.5.Est), viz.

$$\begin{aligned}&\max_{0\le t\le T}\left\{\|\mathcal{T}_\varepsilon v^\varepsilon(t)-V^{\mathrm{ex}}(t)\|^2_{\mathbb{H}_{\mathbb{R}^d}}+\|u^\varepsilon(t)-u(t)\|^2_H\right\}\\&\le C\left\{\|\mathcal{T}_\varepsilon v^\varepsilon_0-V^{\mathrm{ex}}_0\|^2_{\mathbb{H}_{\mathbb{R}^d}}+\|u^\varepsilon_0-u_0\|^2_H+\int_0^T\Delta^{v^\varepsilon}(t)+\Delta^{u^\varepsilon}(t)\,\mathrm{d}t\right\},\end{aligned}\tag{2.3.25}$$

derived in Theorem 2.1.6, we show $\int_0^T\Delta^{u^\varepsilon}(t)+\Delta^{v^\varepsilon}(t)\,\mathrm{d}t\le(\varepsilon+\sqrt{\varepsilon})C$. Using in addition the convergence of the initial values (2.3.2.A4), we obtain the desired estimate (2.3.1a), namely

$$\|\mathcal{T}_\varepsilon v^\varepsilon-V^{\mathrm{ex}}\|_{C([0,T];\mathbb{H}_{\mathbb{R}^d})}+\|u^\varepsilon-u\|_{C([0,T];H)}\le\varepsilon^{1/4}C.$$

Step 3: Gradient estimate. We derive estimate (2.3.1b) for the gradient terms, namely

$$\|\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon)-\nabla_yV^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}+\|\mathcal{T}_\varepsilon(\nabla u^\varepsilon)-[\nabla u+\nabla_yU]^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}\le\varepsilon^{1/4}C.$$

Proof of Theorem 2.3.1. The assumptions (2.3.2.A1) and (2.3.2.A3) on the higher x-regularity of $\mathbb{D}_1$ and the effective solution u imply the improved x-regularity

$$U_i=\sum_{j=1}^d\frac{\partial u_i}{\partial x_j}\cdot z_{ij}\in C([0,T];\mathrm{H}^1(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))$$

for $i=1,\dots,m_1$ according to Lemma 2.1.5 and (2.1.23). For brevity, we set

$$C_F:=\sup_{\substack{(t,x,y)\in[0,T]\times\Omega\times\mathcal{Y}\\ j=1,\dots,d,\,i=1,2}}\{|\mathbb{F}_i(t,x,y,0,0)|+|\partial_{x_j}\mathbb{F}_i(t,x,y,0,0)|\}.\tag{2.3.26}$$

Moreover, let us recall that C_b denotes the uniform bound for the solution $(u^\varepsilon,v^\varepsilon)$ in the space $W_{\mathrm{imp}}(0,T;X)\times W_{\mathrm{imp}}(0,T;X_\varepsilon)$, see (2.1.10). Thanks to the higher x-regularity of the effective solution (u,V), we can employ the gradient folding operators $\widetilde{\mathcal{G}}^0_\varepsilon$ and $\widetilde{\mathcal{G}}^1_\varepsilon$ instead of $\mathcal{G}^0_\varepsilon$ and $\mathcal{G}^1_\varepsilon$, cf. Definition 1.2.7 and 2.3.6.

Step 1: Quantification of Δ^{u^ε}. We derive quantitative estimates for $\Delta^{u^\varepsilon}_1,\dots,\Delta^{u^\varepsilon}_5$. The error $\Delta^{u^\varepsilon}_1$ in (2.1.32) is estimated with Hölder's inequality and Lemma 2.3.8, namely

$$\begin{aligned}\int_0^T|\Delta^{u^\varepsilon}_1|\,\mathrm{d}t&=\int_0^T\bigg|\int_\Omega(F^\varepsilon_1(u^\varepsilon,v^\varepsilon)-u^\varepsilon_t)\cdot(u-\widetilde{\mathcal{G}}^0_\varepsilon(u,U))\\&\qquad-D^\varepsilon_1\nabla u^\varepsilon:\left\{\mathcal{F}_\varepsilon[\nabla u+\nabla_yU]^{\mathrm{ex}}-\nabla\widetilde{\mathcal{G}}^0_\varepsilon(u,U)\right\}\mathrm{d}x\bigg|\,\mathrm{d}t\\&\le C\int_0^T\Big(\|u-\widetilde{\mathcal{G}}^0_\varepsilon(u,U)\|_H+\|\mathcal{F}_\varepsilon[\nabla u+\nabla_yU]^{\mathrm{ex}}-\nabla\widetilde{\mathcal{G}}^0_\varepsilon(u,U)\|_H\Big)\,\mathrm{d}t\\&\le(\varepsilon+\sqrt{\varepsilon})C\Big(\|U\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})))},\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))}\Big).\end{aligned}\tag{2.3.27}$$

Here, we used the growth conditions (1.1.5) and (1.1.8) for D_1^ε and F_1^ε in the first inequality so that the constant C depends additionally on C_b, C_1, and β.

We treat the periodicity defect error $\Delta_2^{u^\varepsilon}$ in (2.1.34), viz.

$$\Delta_2^{u^\varepsilon} = \int_\Omega (F_{\text{eff}}(u,V) - u_t)\cdot u^\varepsilon \,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathcal{Y}} \mathbb{D}_1^{\text{ex}}[\nabla u + \nabla_y U]^{\text{ex}} : \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y, \qquad (2.3.28)$$

with Lemma 2.3.9. Since we have $\mathbb{D}_1^{\text{ex}}[\nabla u+\nabla_y U]^{\text{ex}} \equiv 0$ on $\mathbb{R}^d\backslash\Omega\times\mathcal{Y}$, it suffices to consider the domain of integration $\Omega\times\mathcal{Y}$ on the right-hand side. Recalling the weak formulation (2.1.59) of the u-equations, we find for every $t\in[0,T]$ a two-scale function $\Psi^\varepsilon(t)$ so that $(u^\varepsilon(t),\Psi^\varepsilon(t))\in X\times\mathbb{X}_0$ is an admissible test function. Hence, we have for a.e. $t\in[0,T]$

$$0 \equiv \int_\Omega (F_{\text{eff}}(u,V) - u_t)\cdot u^\varepsilon \,\mathrm{d}x - \int_{\Omega\times\mathcal{Y}} \mathbb{D}_1[\nabla u + \nabla_y U] : [\nabla u^\varepsilon + \nabla_y \Psi^\varepsilon(t)]\,\mathrm{d}x\,\mathrm{d}y. \quad (2.3.29)$$

Since u^ε is of improved time-regularity, the boundedness $\|u^\varepsilon\|_{C([0,T];X)} \le C_b$ holds true. In particular, the continuity of u^ε in time and the estimate $\|\Psi^\varepsilon(t)\|_{\mathbb{X}} \le C\|u\|_X$ imply $\Psi^\varepsilon \in C([0,T];\mathbb{X}_0)$. Subtracting (2.3.29) from (2.3.28) yields with Hölder's inequality as well as the assumptions of higher x-regularity (2.3.2.A1) and (2.3.2.A3)

$$\begin{aligned}
\int_0^T |\Delta_2^{u^\varepsilon}|\,\mathrm{d}t &= \int_0^T \left|\int_{\Omega\times\mathcal{Y}} \mathbb{D}_1[\nabla u + \nabla_y U] : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u^\varepsilon + \nabla_y\Psi^\varepsilon\}]\,\mathrm{d}x\,\mathrm{d}y\right| \mathrm{d}t \\
&\le \int_0^T \|\mathbb{D}_1[\nabla u + \nabla_y U]\|_{\mathrm{L}^2(\mathcal{Y};X^*)^*} \|\,\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u^\varepsilon + \nabla_y\Psi^\varepsilon\}\|_{\mathrm{L}^2(\mathcal{Y};X^*)}\,\mathrm{d}t \\
&\le \int_0^T \|\mathbb{D}_1[\nabla u + \nabla_y U]\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}\varepsilon C\|u^\varepsilon\|_X\,\mathrm{d}t \le (\varepsilon+\sqrt{\varepsilon})C, \qquad (2.3.30)
\end{aligned}$$

where the constant C depends on C_b, $\|\mathbb{D}_1\|_{C([0,T];W^{1,\infty}(\Omega;\mathrm{L}^\infty(\mathcal{Y})))}$, $\|U\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1_{\text{av}}(\mathcal{Y})))}$, and $\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))}$. (Here, we used $\mathrm{L}^2(\mathcal{Y};X^*)^* = \mathrm{L}^2(\mathcal{Y};X) = \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$.)

The third error $\Delta_3^{u^\varepsilon}$ in (2.1.36) is treated with Hölder's inequality as well as Lemma 2.3.3 and 2.3.5:

$$\begin{aligned}
&\int_0^T |\Delta_3^{u^\varepsilon}|\,\mathrm{d}t \\
&= \int_0^T \left|\int_{\mathbb{R}^d\times\mathcal{Y}} (\mathbb{D}_1^{\text{ex}} - \mathcal{T}_\varepsilon D_1^\varepsilon)[\nabla u + \nabla_y U]^{\text{ex}} : \{\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\text{ex}}\}\,\mathrm{d}x\,\mathrm{d}y\right| \mathrm{d}t \\
&\le C(C_b)\int_0^T \|(\mathbb{D}_1^{\text{ex}} - \mathcal{T}_\varepsilon D_1^\varepsilon)[\nabla u + \nabla_y U]^{\text{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}\,\mathrm{d}t \\
&\le C(C_b,\beta)\int_0^T \Big(\|\nabla u + \nabla_y U\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varepsilon^-\times\mathcal{Y})} + \|\mathbb{D}_1 - \mathcal{T}_\varepsilon D_1^\varepsilon\|_{\mathrm{L}^\infty(\Omega_\varepsilon^-\times\mathcal{Y})}\Big)\,\mathrm{d}t \\
&\le (\varepsilon+\sqrt{\varepsilon})C\left(\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))}, \|U\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1_{\text{av}}(\mathcal{Y})))}\right), \qquad (2.3.31)
\end{aligned}$$

where C additionally depends on C_b,β, and $\|\mathbb{D}\|_{C([0,T];W^{1,\infty}(\Omega;\mathrm{L}^\infty(\mathcal{Y})))}$.

The estimation of $\Delta_4^{u^\varepsilon}$ in (2.1.38) is a little more involved. Applying the integral identity (1.2.7) only to the first summand and the averaging formula (2.0.3) for F_{eff} to the second summand yields (pointwise in time)

$$\begin{aligned}
\Delta_4^{u^\varepsilon} &= \int_\Omega [F_1^\varepsilon(u, \mathcal{F}_\varepsilon V) - F_{\mathrm{eff}}(u, V)] \cdot (u^\varepsilon - u)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon\, F_1^\varepsilon(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega\times\mathcal{Y}} \mathbb{F}_1(u, V) \cdot (u^\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Omega_\varepsilon^-\times\mathcal{Y}} \mathcal{T}_\varepsilon\, F_1^\varepsilon(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u) - \mathbb{F}_1(u, V) \cdot (u^\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}y \qquad (2.3.32) \\
&\quad + \int_{\Omega_\varepsilon^+\setminus\Omega_\varepsilon^-\times\mathcal{Y}} \mathcal{T}_\varepsilon\, F_1^\varepsilon(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u) - \mathbb{F}_1^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \cdot [u^\varepsilon - u]^{\mathrm{ex}}\,\mathrm{d}x\,\mathrm{d}y. \\
&\qquad (2.3.33)
\end{aligned}$$

Using the growth condition (1.1.8) for $\mathbb{F}_1$ with C_F from (2.3.26) as well as Lemma 2.3.3 for $u^\varepsilon(t), u(t) \in X$ in (2.3.33) (denoted with $\Delta^{u^\varepsilon}_{4,\Omega_\varepsilon^+}$) gives

$$\left|\Delta^{u^\varepsilon}_{4,\Omega_\varepsilon^+}\right| \leq C(C_b)\|u^\varepsilon - u\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^-)} \leq (\varepsilon + \sqrt{\varepsilon})C(C_b).$$

For now, we set $\mathbb{H}_\varepsilon^- := \mathrm{L}^2(\Omega_\varepsilon^- \times \mathcal{Y})$. Introducing the terms $\pm\mathbb{F}_1(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)$ and $\pm\mathbb{F}_1(u, v) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)$ into (2.3.32) (denoted with $\Delta^{u^\varepsilon}_{4,\Omega_\varepsilon^-}$), applying Hölder's inequality, and recalling the Lipschitz continuity and growth condition for $\mathbb{F}_1$ gives (pointwise in time)

$$\begin{aligned}
\left|\Delta^{u^\varepsilon}_{4,\Omega_\varepsilon^-}\right| &\leq \|\,\mathcal{T}_\varepsilon\, F_1^\varepsilon(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) - \mathbb{F}_1(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V)\|_{\mathbb{H}_\varepsilon^-}\|\,\mathcal{T}_\varepsilon(u^\varepsilon - u)\|_{\mathbb{H}_\varepsilon^-} \\
&\quad + \|\mathbb{F}_1(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) - \mathbb{F}_1(u, V)\|_{\mathbb{H}_\varepsilon^-}\|\,\mathcal{T}_\varepsilon(u^\varepsilon - u)\|_{\mathbb{H}_\varepsilon^-} \\
&\quad + \|\mathbb{F}_1(u, V)\|_{\mathbb{H}_\varepsilon^-}\|\,\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}_\varepsilon^-} \\
&\leq C_*\Big\{\|\,\mathcal{T}_\varepsilon\, F_1^\varepsilon(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V) - \mathbb{F}_1(\mathcal{T}_\varepsilon\, u, \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V)\|_{\mathbb{H}_\varepsilon^-} \qquad (2.3.34) \\
&\qquad + L\left(\|\,\mathcal{T}_\varepsilon\, u - u)\|_{\mathbb{H}_\varepsilon^-} + \|\,\mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V - V\|_{\mathbb{H}_\varepsilon^-}\right) \qquad (2.3.35) \\
&\qquad + \|\,\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}_\varepsilon^-}\Big\}, \qquad (2.3.36)
\end{aligned}$$

where C_* depends on $C_1, \|u\|_{C([0,T];H)}$, $\|u^\varepsilon\|_{C([0,T];H)}$, and $\|V\|_{C([0,T];\mathbb{H})}$. We exploit the improved spatial regularity of $\mathbb{F}_1$ (2.3.2.A1) in (2.3.34) and argue as in Lemma 2.3.5 using the Lipschitz continuity of $\mathbb{F}_1$ with respect to $x \in \Omega$. Moreover, we apply Lemma 2.3.4 to $u^\varepsilon(t), u(t) \in X$ and $V(t) \in \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y}))$ in (2.3.35)–(2.3.36) so that we arrive at

$$\int_0^T |\Delta_4^{u^\varepsilon}|\,\mathrm{d}t \leq (\varepsilon + \sqrt{\varepsilon})C, \qquad (2.3.37)$$

where C depends on C_F, C_*, L, $\|u\|_{\mathrm{L}^2(0,T;X)}$, $\|u^\varepsilon\|_{\mathrm{L}^2(0,T;X)}$, and $\|V\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})))}$.

For the last error term $\Delta_5^{u^\varepsilon}$ in (2.1.40), we have immediately by Lemma 2.3.4

$$\int_0^T \Delta_5^{u^\varepsilon}\,\mathrm{d}t = \int_0^T 2L\|V - \mathcal{T}_\varepsilon\, \mathcal{F}_\varepsilon\, V\|^2_{\mathbb{H}}\,\mathrm{d}t \leq (\varepsilon^2 + \varepsilon)C\|V\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})))}. \qquad (2.3.38)$$

Adding the estimates (2.3.27), (2.3.30)–(2.3.31), and (2.3.37)–(2.3.38) yields

$$\int_0^T \Delta^{u^\varepsilon}\,\mathrm{d}t \leq (\varepsilon + \sqrt{\varepsilon})C. \qquad (2.3.39)$$

Step 2: Quantification of Δ^{v^ε}. We proceed as in Step 1. Applying Lemma 2.3.8 to the folding mismatch $\Delta_1^{v^\varepsilon}$ in (2.1.33) yields

$$\begin{aligned}\int_0^T |\Delta_1^{v^\varepsilon}|\,\mathrm{d}t = \int_0^T \Big| \int_\Omega (F_2^\varepsilon(u^\varepsilon, v^\varepsilon) - v_t^\varepsilon) \cdot (\mathcal{F}_\varepsilon V^{\mathrm{ex}} - \widetilde{\mathcal{G}}_\varepsilon^1 V) \\ - \varepsilon D_2^\varepsilon \nabla v^\varepsilon : \Big[\mathcal{F}_\varepsilon (\nabla_y V)^{\mathrm{ex}} - \varepsilon \nabla(\widetilde{\mathcal{G}}_\varepsilon^1 V)\Big]\,\mathrm{d}x \Big|\,\mathrm{d}t \\ \leq C(C_b, C_1, \beta) \int_0^T \Big(\| \mathcal{F}_\varepsilon V^{\mathrm{ex}} - \widetilde{\mathcal{G}}_\varepsilon^1 V\|_H + \| \mathcal{F}_\varepsilon (\nabla_y V)^{\mathrm{ex}} - \varepsilon \nabla (\widetilde{\mathcal{G}}_\varepsilon^1 V)\|_H \Big)\,\mathrm{d}t \\ \leq (\varepsilon + \sqrt{\varepsilon} C(C_b, C_1, \beta, \|V\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y})))}). \end{aligned} \tag{2.3.40}$$

For the estimation of the periodicity defect $\Delta_2^{v^\varepsilon}$ in (2.1.35), viz.

$$\Delta_2^{v^\varepsilon} = \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - V_t^{\mathrm{ex}}) \cdot \mathcal{T}_\varepsilon v^\varepsilon - \mathbb{D}_2^{\mathrm{ex}} \nabla_y V^{\mathrm{ex}} : \nabla_y (\mathcal{T}_\varepsilon v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y,$$

we use again Lemma 2.3.9. Note, we have $\mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}}) \equiv 0$, $V_t^{\mathrm{ex}} \equiv 0$ and $\mathbb{D}_2^{\mathrm{ex}} \equiv 0$ on $\mathbb{R}^d \backslash \Omega \times \mathcal{Y}$. We recall that $v^\varepsilon \in W_{\mathrm{imp}}(0,T;X_\varepsilon)$ so that $\|v^\varepsilon\|_{C([0,T];X_\varepsilon)} \leq C_b$ according to (2.1.10). Then, Lemma 2.3.9 gives for every $t \in [0,T]$ a function $\Psi^\varepsilon(t) \in \mathbb{X}$ such that $\|\Psi^\varepsilon(t)\|_{\mathbb{X}} \leq \varepsilon C \|v^\varepsilon(t)\|_{X_\varepsilon}$. The continuity of v^ε implies $\Psi^\varepsilon \in C([0,T];\mathbb{X})$. Then, in particular, Ψ^ε is an admissible test function for $(2.0.2.\mathrm{P}_0)_2$, i.e.

$$0 \equiv \int_{\Omega \times \mathcal{Y}} [\mathbb{F}_2(u,V) - V_t] \cdot \Psi^\varepsilon - \mathbb{D}_2 \nabla_y V : \nabla_y \Psi^\varepsilon\,\mathrm{d}x\,\mathrm{d}y. \tag{2.3.41}$$

Hence, the application of Hölder's inequality and Lemma 2.3.9 (cf. Remark 2.3.12) gives

$$\begin{aligned}\int_0^T |\Delta_2^{v^\varepsilon}|\,\mathrm{d}t = \int_0^T \Big| \int_{\Omega \times \mathcal{Y}} [\mathbb{F}_2(u,V) - V_t] \cdot (\mathcal{T}_\varepsilon v^\varepsilon - \Psi^\varepsilon) - \mathbb{D}_2 \nabla_y V : \nabla_y (\mathcal{T}_\varepsilon v^\varepsilon - \Psi^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \Big|\,\mathrm{d}t \\ \leq \int_0^T \|\mathbb{F}_2(u,V) - V_t\|_{\mathrm{L}^2(\mathcal{Y};X^*)^*} \| \mathcal{T}_\varepsilon v^\varepsilon - \Psi^\varepsilon\|_{\mathrm{L}^2(\mathcal{Y};X^*)}\,\mathrm{d}t \\ + \int_0^T \|\mathbb{D}_2 \nabla_y V\|_{\mathrm{L}^2(\mathcal{Y};X^*)^*} \|\nabla_y (\mathcal{T}_\varepsilon v^\varepsilon - \Psi^\varepsilon)\|_{\mathrm{L}^2(\mathcal{Y};X^*)}\,\mathrm{d}t \\ \leq \int_0^T \| |\mathbb{D}_2 \nabla_y V| + |\mathbb{F}_2(u,V)| + |V_t| \|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))} \| \mathcal{T}_\varepsilon v^\varepsilon - \Psi^\varepsilon\|_{\mathrm{H}^1(Y;X^*)}\,\mathrm{d}t \\ \leq (\varepsilon + \sqrt{\varepsilon}) C, \end{aligned} \tag{2.3.42}$$

where the constant C depends on C_F, C_b, $\|\mathbb{D}_2\|_{C([0,T];W^{1,\infty}(\Omega;\mathrm{L}^\infty(\mathcal{Y})))}$, $\|V\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y})))}$, and $\|V_t\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})))}$.

The approximation errors $\Delta_3^{v^\varepsilon}$–$\Delta_5^{v^\varepsilon}$ in (2.1.36)–(2.1.40) are estimated easily by using Lemma 2.3.3, 2.3.4, and 2.3.5:

$$\begin{aligned}\int_0^T |\Delta_3^{v^\varepsilon}|\,\mathrm{d}t = \int_0^T \Big| \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathbb{D}_2^{\mathrm{ex}} - \mathcal{T}_\varepsilon D_2^\varepsilon) \nabla_y V^{\mathrm{ex}} : \nabla_y (\mathcal{T}_\varepsilon v^\varepsilon - V^{\mathrm{ex}})\,\mathrm{d}x\,\mathrm{d}y \Big|\,\mathrm{d}t \\ \leq C(C_b) \int_0^T \|\nabla_y V\|_{\mathrm{L}^2(\Omega \backslash \Omega_\varepsilon^- \times \mathcal{Y})} + \|(\mathbb{D}_2 - \mathcal{T}_\varepsilon D_2^\varepsilon)\|_{\mathrm{L}^\infty(\Omega_\varepsilon^- \times \mathcal{Y})}\,\mathrm{d}t \\ \leq (\varepsilon + \sqrt{\varepsilon}) C, \end{aligned} \tag{2.3.43}$$

$$\begin{aligned}
\int_0^T |\Delta_4^{v^\varepsilon}|\,\mathrm{d}t &= \int_0^T \left| \int_{\mathbb{R}^d \times \mathcal{Y}} [\mathcal{T}_\varepsilon\, F_2^\varepsilon(u^{\mathrm{ex}}, V^{\mathrm{ex}}) - \mathbb{F}_2^{\mathrm{ex}}(u^{\mathrm{ex}}, V^{\mathrm{ex}})] \cdot (\mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}})\,\mathrm{d}x\,\mathrm{d}y \right| \mathrm{d}t \\
&\le C_b \int_0^T 2\|C_1(1 + |u| + |V|)\|_{\mathrm{L}^2(\Omega \setminus \Omega_\varepsilon^- \times \mathcal{Y})} \\
&\qquad\quad + \|\, \mathcal{T}_\varepsilon\, F_2^\varepsilon(u, V) - \mathbb{F}_2(u, V)\|_{\mathrm{L}^2(\Omega_\varepsilon^- \times \mathcal{Y})}\,\mathrm{d}t \\
&\le (\varepsilon + \sqrt{\varepsilon})C, \qquad (2.3.44)\\
\int_0^T |\Delta_5^{v^\varepsilon}|\,\mathrm{d}t &= \int_0^T 2L\|\, \mathcal{T}_\varepsilon\, u - u\|_{\mathbb{H}}^2\,\mathrm{d}t \le (\varepsilon^2 + \varepsilon)C\|u\|_{\mathrm{L}^2(0,T;X)}^2, \qquad (2.3.45)
\end{aligned}$$

where C depends on C_F, C_b, $\|\mathbb{D}_2\|_{C([0,T];W^{1,\infty}(\Omega;\mathrm{L}^\infty(\mathcal{Y})))}$, and $\|V\|_{\mathrm{L}^2(\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y})))}$. Overall (2.3.40)–(2.3.45) give

$$\int_0^T \Delta^{v^\varepsilon}\,\mathrm{d}t \le (\varepsilon + \sqrt{\varepsilon})C.$$

Combined with the estimation of Δ^{u^ε} (2.3.39), we finish the proof of (2.3.1a) by inserting the assumption on the initial values (2.3.2.A4), i.e. $\|\, \mathcal{T}_\varepsilon\, v_0^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u_0^\varepsilon - u_0\|_H \le \varepsilon^{1/4}c$, into (2.3.25). Taking the square root in (2.3.25) gives the convergence rate $\varepsilon^{1/4}$.

Step 3: Derivation of the gradient estimate (2.3.1b). Adding the Gronwall-type estimates (2.1.55) and (2.1.62) for $\mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}}$ and $u^\varepsilon - u$ in the proof of (2.0.5.Est) gives

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left\{\|\, \mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|u^\varepsilon - u\|_H^2\right\} \\
&\le -\alpha\left\{\|\, \mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon) - \nabla_y V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|[\nabla u + \nabla_y U]^{\mathrm{ex}} - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\|_{\mathbb{H}_{\mathbb{R}^d}}^2\right\} \\
&\quad + \tfrac{3}{2}L\left\{\|\, \mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|u^\varepsilon - u\|_H^2\right\} + \Delta^{v^\varepsilon} + \Delta^{u^\varepsilon}.
\end{aligned}$$

Integrating over $[0, T]$ and exploiting (2.3.1a) as well as the results of Step 1–2 yields

$$\begin{aligned}
&\alpha\left\{\|\, \mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon) - \nabla_y V^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}^2 + \|[\nabla u + \nabla_y U]^{\mathrm{ex}} - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}^2\right\} \\
&\le \tfrac{1}{2}\left\{\|\, \mathcal{T}_\varepsilon\, v_0^\varepsilon - V_0^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|u_0^\varepsilon - u_0\|_H^2 - \|\, \mathcal{T}_\varepsilon\, v^\varepsilon(T) - V(T)^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 - \|u^\varepsilon(T) - u(T)\|_H^2\right\} \\
&\quad + \tfrac{3}{2}L\left\{\|\, \mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}^2 + \|u^\varepsilon - u\|_{\mathrm{L}^2(0,T;\mathbb{H}_{\mathbb{R}^d})}^2\right\} + \int_0^T \Delta^{v^\varepsilon} + \Delta^{u^\varepsilon}\,\mathrm{d}t \\
&\le \tfrac{1}{2}(2 + 3LT)\left\{\|\, \mathcal{T}_\varepsilon\, v^\varepsilon - V^{\mathrm{ex}}\|_{C([0,T];\mathbb{H}_{\mathbb{R}^d})}^2 + \|u^\varepsilon - u\|_{C([0,T];H)}^2\right\} + \int_0^T \Delta^{v^\varepsilon} + \Delta^{u^\varepsilon}\,\mathrm{d}t \\
&\le (\varepsilon + \sqrt{\varepsilon})C.
\end{aligned}$$

Hence, (2.3.1b) is shown and the proof of Theorem 2.3.1 is finished. □

2.3.5 Proof of Main Theorem IIIb

We prove an error estimate for the solutions $(u^\varepsilon, v^\varepsilon)$ and (u, V) of (2.0.1.P$_\varepsilon$) and (2.0.2.P$_0$), respectively, without assuming a priori improved time-regularity for $(u^\varepsilon, v^\varepsilon)$. By the assumptions (2.3.2.A0)–(2.3.2.A2), we have $u^\varepsilon \in W(0, T, X)$ and $v^\varepsilon \in W(0, T; X_\varepsilon)$, only. In the proof of Main Theorem IIIa, we exploit the uniform boundedness of u_t^ε and v_t^ε in

$C([0,T];H)$ (via improved-time regularity by (2.3.2.A4)) to estimate the folding mismatch errors, viz.

$$\Delta_1^{u^\varepsilon} := \int_\Omega (F_1^\varepsilon(u^\varepsilon,v^\varepsilon)-u_t^\varepsilon)\cdot(u-\widetilde{\mathcal{G}}_\varepsilon^0(u,U))$$
$$\qquad - D_1^\varepsilon\nabla u^\varepsilon : \{\mathcal{F}_\varepsilon[\nabla u+\nabla_y U]^{\mathrm{ex}} - \nabla\,\widetilde{\mathcal{G}}_\varepsilon^0(u,U)\}\,\mathrm{d}x, \tag{2.3.46}$$
$$\Delta_1^{v^\varepsilon} := \int_\Omega (F_2^\varepsilon(u^\varepsilon,v^\varepsilon)-v_t^\varepsilon)\cdot[\mathcal{F}_\varepsilon V^{\mathrm{ex}}-\widetilde{\mathcal{G}}_\varepsilon^1 V] - \varepsilon D_2^\varepsilon\nabla v^\varepsilon : \Big[\mathcal{F}_\varepsilon(\nabla_y V)^{\mathrm{ex}} - \varepsilon\nabla(\widetilde{\mathcal{G}}_\varepsilon^1 V)\Big]\,\mathrm{d}x.$$

For general solutions $(u^\varepsilon,v^\varepsilon)$ this uniform boundedness does not hold. However, we can proceed as in Main Theorem II and approximate $(u^\varepsilon,v^\varepsilon)$ with a regularized solution $(u^{\varepsilon,\delta},v^{\varepsilon,\delta})$ of improved time-regularity. As before, the folding mismatch is estimated (for e.g. $\gamma=1$) via

$$|\Delta_1^{v^{\varepsilon,\delta}}| \leq C(\delta)\Big\{\|\,\mathcal{F}_\varepsilon V^\delta - \widetilde{\mathcal{G}}_\varepsilon^1 V^\delta\|_H + \|\,\mathcal{F}_\varepsilon(\nabla_y V^\delta) - \varepsilon\nabla(\widetilde{\mathcal{G}}_\varepsilon^1 V^\delta)\|_H\Big\} \leq \delta^{-1}(\varepsilon+\sigma\sqrt{\varepsilon})C$$

with $\sigma=0$ if $\Omega_\varepsilon^+=\Omega_\varepsilon^-$ and $\sigma=1$ otherwise. Then, choosing $\delta=\delta(\varepsilon)$ suitably, we obtain the convergence rate $\varepsilon^{1/3}$ in the case $\sigma=0$ respective $\varepsilon^{1/6}$ in the case $\sigma=1$. Recall that the effective solution (u,V) satisfies higher regularity by (2.3.2.A2) in any case.

Proof of Theorem 2.3.2. We regularize the given initial values $(u_0^\varepsilon,v_0^\varepsilon)$ via $u_0^{\varepsilon,\delta} := \mathcal{R}_{\varepsilon,\delta}^0 u_0^\varepsilon$ and $v_0^{\varepsilon,\delta} := \mathcal{R}_{\varepsilon,\delta}^1 v_0^\varepsilon$ as in (2.2.15a)–(2.2.15b), i.e.

$$u_0^{\varepsilon,\delta} - \operatorname{div}(\delta^2 D_1^\varepsilon(0)\nabla u_0^{\varepsilon,\delta}) = u_0^\varepsilon \quad\text{and}\quad v_0^{\varepsilon,\delta} - \operatorname{div}(\delta^2\varepsilon^2 D_2^\varepsilon(0)\nabla v_0^{\varepsilon,\delta}) = v_0^\varepsilon. \tag{2.3.47}$$

Testing the weak formulation of $(2.3.47)_1$ with $\varphi = u_0^{\varepsilon,\delta}-u_0^\varepsilon$, exploiting the ellipticity (1.1.5) of D_1^ε, and applying Young's inequality gives

$$\begin{aligned}\int_\Omega (u_0^{\varepsilon,\delta}-u_0^\varepsilon)^2\,\mathrm{d}x &= \delta^2\int_\Omega -D_1^\varepsilon\nabla u_0^{\varepsilon,\delta} : \nabla(u_0^{\varepsilon,\delta}-u_0^\varepsilon)\,\mathrm{d}x\\ &\leq \delta^2\Big\{-\alpha\|\nabla u_0^{\varepsilon,\delta}\|_H^2 + \beta\|\nabla u_0^{\varepsilon,\delta}\|_H\|\nabla u_0^\varepsilon\|_H\Big\}\\ &\leq \delta^2\Big\{-\alpha\|\nabla u_0^{\varepsilon,\delta}\|_H^2 + \beta\tfrac{\alpha}{\beta}\|\nabla u_0^{\varepsilon,\delta}\|_H^2 + \beta\tfrac{\beta}{\alpha}\|\nabla u_0^\varepsilon\|_H^2\Big\}.\end{aligned}$$

Proceeding analogously with $(2.3.47)_2$ yields

$$\|u_0^{\varepsilon,\delta}-u_0^\varepsilon\|_H \leq \delta C(\alpha,\beta)\|u_0^\varepsilon\|_X \quad\text{as well as}\quad \|v_0^{\varepsilon,\delta}-v_0^\varepsilon\|_H \leq \delta C(\alpha,\beta)\|v_0^\varepsilon\|_{X_\varepsilon}.$$

Using the boundedness $\|u_0^\varepsilon\|_X + \|v_0^\varepsilon\|_{X_\varepsilon} \leq c$ by assumption (2.3.4), we obtain

$$\|u_0^{\varepsilon,\delta}-u_0^\varepsilon\|_H + \|v_0^{\varepsilon,\delta}-v_0^\varepsilon\|_H \leq \delta C_*. \tag{2.3.48}$$

Now, let $u^{\varepsilon,\delta}\in W_{\mathrm{imp}}(0,T;X)$ and $v^{\varepsilon,\delta}\in W_{\mathrm{imp}}(0,T;X_\varepsilon)$ denote the associated regularized solutions with $u^{\varepsilon,\delta}(0)=u_0^{\varepsilon,\delta}$ and $v^{\varepsilon,\delta}(0)=v_0^{\varepsilon,\delta}$. Therefore, Lemma 2.2.4 yields

$$\begin{aligned}&\|u^{\varepsilon,\delta}-u^\varepsilon\|_{C([0,T];H)} + \|\nabla u^{\varepsilon,\delta}-\nabla u^\varepsilon\|_{\mathrm{L}^2(0,T;H)}\\ &+\|v^{\varepsilon,\delta}-v^\varepsilon\|_{C([0,T];H)} + \|\nabla v^{\varepsilon,\delta}-\nabla v^\varepsilon\|_{\mathrm{L}^2(0,T;H)} \leq \delta C_*.\end{aligned} \tag{2.3.49}$$

From the equations (2.3.47) and (2.3.48), we obtain in particular

$$\|\operatorname{div}(D_1^\varepsilon(0)\nabla u_0^{\varepsilon,\delta})\|_H + \|\operatorname{div}(\varepsilon^2 D_2^\varepsilon(0)\nabla v_0^{\varepsilon,\delta})\|_H \leq \tfrac{C}{\delta}.$$

Reviewing the estimates in the proof Theorem 1.1.2 on the existence of solutions and Proposition 1.1.3 on improved time-regularity gives

$$\|u_t^{\varepsilon,\delta}\|_{C([0,T];H)} + \|v_t^{\varepsilon,\delta}\|_{C([0,T];H)} \le \tfrac{C}{\delta}. \tag{2.3.50}$$

Therefore the folding mismatch (2.3.46) is bounded by Lemma 2.3.8 and (2.3.50) via

$$\int_0^T |\Delta_1^{u^{\varepsilon,\delta}}(t)| + |\Delta_1^{v^{\varepsilon,\delta}}(t)| \,\mathrm{d}t \le \left(1 + \tfrac{1}{\delta}\right)(\varepsilon + \sigma\sqrt{\varepsilon})\, C.$$

The remaining error terms do not rely on the improved time-regularity of $(u^\varepsilon, v^\varepsilon)$ so that

$$i = 2,3,4,5: \quad \int_0^T |\Delta_i^{u^{\varepsilon,\delta}}(t)| + |\Delta_i^{v^{\varepsilon,\delta}}(t)| \,\mathrm{d}t \le (\varepsilon + \sqrt{\varepsilon})\, C.$$

Based on estimate (2.0.5.Est) and $\| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}} + \|u_0^\varepsilon - u_0\|_H \le \varepsilon^{1/2} c$ by assumption (2.3.4), we arrive at

$$\begin{aligned}
&\max_{0\le t\le T} \left\{ \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V^{\mathrm{ex}}(t)\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|u^\varepsilon(t) - u(t)\|_H^2 \right\} \\
&\le \|v^{\varepsilon,\delta} - v^\varepsilon\|_{C([0,T];H)}^2 + \|u^{\varepsilon,\delta} - u^\varepsilon\|_{C([0,T];H)}^2 \\
&\quad + C \left\{ \| \mathcal{T}_\varepsilon v_0^{\varepsilon,\delta} \pm \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}}\|_{\mathbb{H}_{\mathbb{R}^d}}^2 + \|u_0^{\varepsilon,\delta} \pm u_0^\varepsilon - u_0\|_H^2 + \int_0^T \Delta^{v^{\varepsilon,\delta}}(t) + \Delta^{u^{\varepsilon,\delta}}(t)\,\mathrm{d}t \right\} \\
&\le \left(\delta^2 + \varepsilon + \sqrt{\varepsilon} + \tfrac{\varepsilon}{\delta} + \sigma\tfrac{\sqrt{\varepsilon}}{\delta}\right) C.
\end{aligned}$$

Let $\sigma = 1$ and $\delta = \varepsilon^x$. We choose x such that $\varepsilon^{2x} = \varepsilon^{1/2-x}$ which is $x = 1/6$. Inserting $\delta = \varepsilon^{1/6}$ and taking the square root of the latter estimate yields the desired convergence rate $\varepsilon^{1/6}$ on arbitrary domains Ω.

Following again Step 3 of the proof of Theorem 2.3.1 and using (2.3.49) gives the corresponding estimates for the gradient terms. □

2.3.6 Interior estimates for slow diffusion only

In the case that our original system (2.0.1.P$_\varepsilon$) contains only slowly diffusing species, we find the better convergence rate $\varepsilon^{1/2}$ in the interior of the domain Ω. This subsection is devoted to the study of the reduced problem

$$v_t^\varepsilon = \mathrm{div}(\varepsilon^2 D_2^\varepsilon \nabla v^\varepsilon) + F_2^\varepsilon(v^\varepsilon) \qquad \text{in } [0,T]\times\Omega \tag{2.3.51}$$

with initial and boundary conditions as before. The interior of the domain Ω means the following: We define for $\varrho > 0$ the subset $\Omega_\varrho \subsetneqq \Omega$ via

$$\Omega_\varrho := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \varrho\}.$$

By employing a suitable cut-off function, we can improve the estimations for the error terms Δ^{v^ε} from $\sqrt{\varepsilon}$ to ε. Away from the boundary $\partial\Omega$, the condition $\Omega_\varepsilon^- = \Omega_\varepsilon^+$ as well as the boundary conditions are negligible and we can set $\sigma = \eta = 0$ in (2.3.3). For classically diffusing species u^ε this approach via cut-off function does not seem to apply. However, we expect that the better rate $\varepsilon^{1/2}$ (up to the boundary $\partial\Omega$) can be proved for the u^ε-equations, decoupled from v^ε, using a different approach. Throughout this subsection, v^ε denotes the solution of the original problem (2.3.51) and V denotes the solution of the corresponding limit problem $(2.0.2.\mathrm{P}_0)_2$ with $\mathbb{F}_2(V)$ being independent of u.

Theorem 2.3.13. *Let the assumptions* (2.3.2) *be satisfied. If the initial values additionally satisfy*

$$\forall\,\varrho>0\ \exists\,c_\varrho\geq 0\ \forall\,\varepsilon<\varrho/(2\sqrt{d}):\quad \|\mathcal{T}_\varepsilon\, v_0^\varepsilon - V_0\|_{\mathrm{L}^2(\Omega_\varrho\times\mathcal{Y})}\leq \varepsilon^{1/2}c_\varrho, \tag{2.3.52}$$

then there exists a constant $C_\varrho\geq 0$ independent of ε such that it holds for all $\varepsilon<\varrho/(4\sqrt{d})$

$$\|\mathcal{T}_\varepsilon\, v^\varepsilon - V\|_{C([0,T];\mathrm{L}^2(\Omega_\varrho\times\mathcal{Y}))}+\|\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon)-\nabla_y V\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_\varrho\times\mathcal{Y}))}\leq \varepsilon^{1/2}C_\varrho.$$

Before we give the proof, note that the improved folding mismatch $\Delta_1^{v^\varepsilon}\sim O(\varepsilon)$ on Ω_ϱ also implies the better convergence rate $\varepsilon^{1/3}$ (compared to $\varepsilon^{1/6}$) for more general initial conditions (as in Main Theorem IIIb). Note that the following estimate is not exclusively restricted to the interior domain Ω_ϱ. If Ω satisfies $\Omega_{\varepsilon_k}^-=\Omega_{\varepsilon_k}^+$ for a suitable sequence $\varepsilon_k\to 0$, we can set $\sigma=0$ in (2.3.3) and, then, the error $\varepsilon^{1/3}$ holds up to the boundary.

Corollary 2.3.14. *Let the assumptions* (2.3.2.A0)–(2.3.2.A3) *hold and let the initial values satisfy* (2.3.4). *Then, there exists a constant $C_\varrho\geq 0$ independent of $\varepsilon>0$ such that it holds for all $\varepsilon<\varrho/(2\sqrt{d})$*

$$\|\mathcal{T}_\varepsilon\, v^\varepsilon - V\|_{C([0,T];\mathrm{L}^2(\Omega_\varrho\times\mathcal{Y}))}+\|\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon)-\nabla_y V\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_\varrho\times\mathcal{Y}))}\leq \varepsilon^{1/3}C_\varrho.$$

Proof. Choose $\sigma=0$ and $x=1/3$ at the end of the proof for Theorem 2.3.2. □

Proof of Theorem 2.3.13. For $\varrho>0$ fixed, we construct a nonnegative cut-off function $\vartheta_\varrho\in C_c^\infty(\Omega)$ such that

$$\vartheta_\varrho\equiv 1 \text{ on } \Omega_{2\varrho}\quad\text{and}\quad \vartheta_\varrho\equiv 0 \text{ on } \Omega\backslash\Omega_\varrho.$$

Let $\varepsilon<\varrho/(2\sqrt{d})$. We emphasize that Ω_ε^- is strictly contained in Ω_ϱ for all $\varepsilon<\varrho/(2\sqrt{d})$ and it holds

$$\operatorname{supp}(\mathcal{T}_\varepsilon\,\vartheta_\varrho v)\subset\overline{\Omega_\varepsilon^-}\times\mathcal{Y}\quad\text{and}\quad\operatorname{supp}(\vartheta_\varrho v)\subset\overline{\Omega_\varepsilon^-}\quad\text{for all } v\in\mathrm{L}^1(\Omega). \tag{2.3.53}$$

Hence, we can reduce all following calculations to Ω_ε^-, where the unfolding operator is exact. In the proof of Theorem 2.1.6, we test $(2.0.1.\mathrm{P}_\varepsilon)_2$ with $\varphi=\vartheta_\varrho^2 v^\varepsilon-\mathcal{G}_\varepsilon^1(\vartheta_\varrho^2 V)$ in (2.1.45), viz.

$$\begin{aligned}\int_\Omega & v_t^\varepsilon\cdot[\vartheta_\varrho^2 v^\varepsilon-\mathcal{G}_\varepsilon^1(\vartheta_\varrho^2 V)]\,\mathrm{d}x\\ &=\int_\Omega -D_2^\varepsilon\varepsilon\nabla v^\varepsilon:\varepsilon\nabla[\vartheta_\varrho^2 v^\varepsilon-\mathcal{G}_\varepsilon^1(\vartheta_\varrho^2 V)]+F_2^\varepsilon(v^\varepsilon)\cdot[\vartheta_\varrho^2 v^\varepsilon-\mathcal{G}_\varepsilon^1(\vartheta_\varrho^2 V)]\,\mathrm{d}x.\end{aligned}$$

In the same manner, we test $(2.0.2.\mathrm{P}_0)_2$ with $\Phi=\vartheta_\varrho^2 V$ in (2.1.49), viz.

$$\int_{\Omega\times\mathcal{Y}} V_t\cdot\vartheta_\varrho^2 V\,\mathrm{d}x\,\mathrm{d}y=\int_{\Omega\times\mathcal{Y}} -\mathbb{D}_2\nabla_y V:\nabla_y(\vartheta_\varrho^2 V)+\mathbb{F}_2(V)\cdot\vartheta_\varrho^2 V\,\mathrm{d}x\,\mathrm{d}y.$$

And for all further reformulations throughout the proof of Theorem 2.1.6, we incorporate the cut-off function ϑ_ϱ. Thus, we obtain the estimate (cf. (2.0.5.Est) or (2.3.25))

$$\| \mathcal{T}_\varepsilon(\vartheta_\varrho v^\varepsilon) - \vartheta_\varrho V\|^2_{C([0,T];\mathbb{H})} \leq C \left\{ \| \mathcal{T}_\varepsilon(\vartheta_\varrho v_0^\varepsilon) - \vartheta_\varrho V_0\|^2_{\mathbb{H}} + \int_0^T \Delta_\varrho^{v^\varepsilon}(t)\,\mathrm{d}t \right\}, \tag{2.3.54}$$

where the error term $\Delta_\varrho^{v^\varepsilon}$ depends on ϑ_ϱ, too, and new unfolding errors occur with respect to $\mathcal{T}_\varepsilon\,\vartheta_\varrho$. More precisely, it is the folding mismatch

$$\begin{aligned}\Delta_{\varrho,1}^{v^\varepsilon} := \int_{\Omega_\varepsilon^-} &(F_2^\varepsilon(v^\varepsilon) - v_t^\varepsilon) \cdot [\mathcal{F}_\varepsilon(\vartheta_\varrho^2 V) - \mathcal{G}_\varepsilon^1(\vartheta_\varrho^2 V)] \\ &- \varepsilon D_2^\varepsilon \nabla v^\varepsilon : \left[\mathcal{F}_\varepsilon(\nabla_y[\vartheta_\varrho^2 V]) - \varepsilon\nabla(\mathcal{G}_\varepsilon^1[\vartheta_\varrho^2 V])\right] \mathrm{d}x,\end{aligned} \tag{2.3.55}$$

the new unfolding error

$$\begin{aligned}\Delta_{\varrho,1a}^{v^\varepsilon} := \int_{\Omega_\varepsilon^-\times\mathcal{Y}} &\mathcal{T}_\varepsilon[F_2^\varepsilon(v^\varepsilon) - v_t^\varepsilon] \cdot (\mathcal{T}_\varepsilon\,\vartheta_\varrho - \vartheta_\varrho)V \\ &- \mathcal{T}_\varepsilon\, D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon\, v^\varepsilon) : [\nabla_y(\mathcal{T}_\varepsilon\,\vartheta_\varrho)\,\mathcal{T}_\varepsilon(\vartheta_\varrho v^\varepsilon) + (\mathcal{T}_\varepsilon\,\vartheta_\varrho - \vartheta)\nabla_y(\vartheta_\varrho V)] \\ &+ \mathcal{T}_\varepsilon\, D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon\,\vartheta_\varrho)\,\mathcal{T}_\varepsilon\, v^\varepsilon : \nabla_y[\mathcal{T}_\varepsilon(\vartheta_\varrho v^\varepsilon) - \vartheta_\varrho V]\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{2.3.56}$$

the periodicity defect

$$\Delta_{\varrho,2}^{v^\varepsilon} := \int_{\Omega_\varepsilon^-\times\mathcal{Y}} \vartheta_\varrho[\mathbb{F}_2(V) - V_t]\cdot\mathcal{T}_\varepsilon(\vartheta_\varrho v^\varepsilon) - \mathbb{D}_2\nabla_y(\vartheta_\varrho V) : \nabla_y(\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon])\,\mathrm{d}x\,\mathrm{d}y, \tag{2.3.57}$$

and the approximations errors

$$\Delta_{\varrho,3}^{v^\varepsilon} := \int_{\Omega_\varepsilon^-\times\mathcal{Y}} (\mathbb{D}_2 - \mathcal{T}_\varepsilon\, D_2^\varepsilon)\nabla_y(\vartheta_\varrho V) : \nabla_y(\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon] - \vartheta_\varrho V)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.3.58}$$

$$\Delta_{\varrho,4}^{v^\varepsilon} := \int_{\Omega_\varepsilon^-\times\mathcal{Y}} [\mathcal{T}_\varepsilon\,\vartheta_\varrho\,\mathcal{T}_\varepsilon\, F_2^\varepsilon(V) - \vartheta_\varrho\mathbb{F}_2(V)]\cdot(\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon] - \vartheta_\varrho V)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.3.59}$$

Note that the unfolding error $\Delta_5^{v^\varepsilon}$ due to coupling vanishes. To obtain the new unfolding error $\Delta_{\varrho,1a}^{v^\varepsilon}$, we use (cf. Step 1(a) to reformulate $(2.0.1.\mathrm{P}_\varepsilon)_2$)

$$\begin{aligned}&\mathcal{T}_\varepsilon\,\vartheta_\varrho(\mathcal{T}_\varepsilon\, v^\varepsilon)_t = \mathcal{T}_\varepsilon(\vartheta_\varrho v_t^\varepsilon) = (\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon])_t, \\ &\nabla_y(\vartheta_\varrho^2 V) = \vartheta_\varrho\nabla_y(\vartheta_\varrho V) = \vartheta_\varrho^2\nabla_y V, \\ &\mathcal{T}_\varepsilon(\varepsilon\nabla[\vartheta_\varrho^2 v^\varepsilon]) = \nabla_y(\mathcal{T}_\varepsilon[\vartheta_\varrho^2 v^\varepsilon]) = \nabla_y(\mathcal{T}_\varepsilon\,\vartheta_\varrho)\,\mathcal{T}_\varepsilon(\vartheta_\varrho v^\varepsilon) + \mathcal{T}_\varepsilon\,\vartheta_\varrho\nabla_y(\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon]), \\ &\nabla_y(\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon]) = \nabla_y(\mathcal{T}_\varepsilon\,\vartheta_\varrho)\,\mathcal{T}_\varepsilon\, v^\varepsilon + \mathcal{T}_\varepsilon\,\vartheta_\varrho\nabla_y(\mathcal{T}_\varepsilon\, v^\varepsilon).\end{aligned} \tag{2.3.60}$$

The estimation of the error terms follows in principle as in the proof of Main Theorem IIIa. We begin with the new unfolding error $\Delta_{\varrho,1a}^{v^\varepsilon}$ in (2.3.56). First, we note

$$\begin{aligned}&\|\vartheta_\varrho - \mathcal{T}_\varepsilon\,\vartheta_\varrho\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})} \leq \varepsilon C\|\nabla\vartheta_\varrho\|_{\mathrm{L}^\infty(\Omega)}, \\ &\|\nabla_y(\mathcal{T}_\varepsilon\,\vartheta_\varrho)\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})} = \|\mathcal{T}_\varepsilon(\varepsilon\nabla\vartheta_\varrho)\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})} \leq \varepsilon\|\nabla\vartheta_\varrho\|_{\mathrm{L}^\infty(\Omega)}\end{aligned} \tag{2.3.61}$$

thanks to the smoothness of ϑ_ϱ and Lemma 2.3.5. With this as well as the boundedness of the given data and solutions, it holds

$$|\Delta_{\varrho,1a}^{v^\varepsilon}| \leq C(\beta, C_1, C_b)\left\{\|\mathcal{T}_\varepsilon\,\vartheta_\varrho - \vartheta_\varrho\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})} + \|\nabla_y(\mathcal{T}_\varepsilon\,\vartheta_\varrho)\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})}\right\} \leq \varepsilon C_\varrho(t).$$

In particular, the constant $C_\varrho(t)$ depends on the norm $\|\vartheta_\varrho\|_{W^{1,\infty}(\Omega)}$.

Recalling (2.3.53), we can apply all preparatory estimates in Subsection 2.3.1 and 2.3.2 with $\sigma = 0$. With this, the estimation of all error terms but the periodicity defect gives

$$\text{for } i = 1, 1a, 3, 4: \quad \int_0^T |\Delta^{v^\varepsilon}_{\varrho,i}(t)| \, \mathrm{d}t \le \varepsilon C_\varrho.$$

We aim to apply Lemma 2.3.10 instead of Lemma 2.3.9 to the periodicity defect (2.3.57) in order to improve the estimate for $\Delta^{v^\varepsilon}_{\varrho,2}$ from $(\varepsilon + \sqrt{\varepsilon})C$ to εC. We point out that the estimation of the periodicity defect in Lemma 2.3.10 does not depend on whether Ω is an exact union of cells $\varepsilon(\lambda_i + Y)$ or not.

Let $\Psi^\varepsilon \in \mathrm{L}^2(\Omega^-_\varepsilon; \mathrm{H}^1(\mathcal{Y}))$ be as in Lemma 2.3.10 ($\gamma = 1$) for $\vartheta_\varrho v^\varepsilon \in \mathrm{H}^1(\Omega^-_\varepsilon)$ and, thus, $\vartheta_\varrho \Psi^{\varepsilon,\mathrm{ex}} \in \mathbb{X}$ is an admissible test function for the V-equations (cf. (2.0.2.P$_0$) or (2.3.41)) such that

$$0 \equiv \int_{\Omega^-_\varepsilon \times \mathcal{Y}} \vartheta_\varrho[\mathbb{F}_2(V) - V_t] \cdot \Psi^\varepsilon - \mathbb{D}_2 \nabla_y(\vartheta_\varrho V) : \nabla_y \Psi^\varepsilon \, \mathrm{d}x \, \mathrm{d}y.$$

Hence, it is

$$\begin{aligned} \Delta^{v^\varepsilon}_{\varrho,2} = \int_{\Omega^-_\varepsilon \times \mathcal{Y}} & \vartheta_\varrho[\mathbb{F}_2(V) - V_t] \cdot (\mathcal{T}_\varepsilon(\vartheta_\varrho v^\varepsilon) - \Psi^\varepsilon) \\ & - \mathbb{D}_2 \nabla_y(\vartheta_\varrho V) : (\nabla_y(\mathcal{T}_\varepsilon[\vartheta_\varrho v^\varepsilon]) - \Psi^\varepsilon) \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{2.3.62}$$

Thanks to $\vartheta_\varrho[\mathbb{F}_2(V) - V_t] = 0$ and $\mathbb{D}_2 \nabla_y(\vartheta_\varrho V) = 0$ almost everywhere on $\partial\Omega \times \mathcal{Y}$, Lemma 2.3.10 yields the desired estimate $|\Delta^{v^\varepsilon}_{\varrho,2}(t)| \le \varepsilon C_\varrho(t)$. Since all involved terms are integrable on $(0,T)$, we obtain $\int_0^T |\Delta^{v^\varepsilon}_{\varrho,2}(t)| \, \mathrm{d}t \le \varepsilon C_\varrho$.

Using further $0 \le \vartheta_\varrho \le 1$ and the convergence of the initial values in (2.3.52), we can estimate the right-hand side of (2.3.54) by εC_ϱ. Estimating the left-hand side of (2.3.54) from below by replacing Ω with $\Omega_{2\varrho}$, yields overall

$$\| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{C([0,T]; \mathrm{L}^2(\Omega_{2\varrho} \times \mathcal{Y}))} \le \varepsilon^{1/2} C_\varrho.$$

Letting $\varepsilon < \varrho/(4\sqrt{d})$ instead of $\varepsilon < \varrho/(2\sqrt{d})$ gives the desired estimate on Ω_ϱ. We finish the proof of Theorem 2.3.13 by repeating the gradient estimates for v^ε in Step 3(a) of the proof of Theorem 2.3.1. □

2.3.7 On the choice of the initial values

We evaluate for which choice of initial values assumption (2.3.2.A4) in Main Theorem IIIa holds, namely

$$\begin{aligned} & \|u^\varepsilon_0\|_H + \| \operatorname{div}(D^\varepsilon_1(0) \nabla u^\varepsilon_0) \|_H + \|v^\varepsilon_0\|_H + \| \operatorname{div}(\varepsilon^2 D^\varepsilon_2(0) \nabla v^\varepsilon_0) \|_H \le c, \\ & \|u^\varepsilon_0 - u_0\|_H + \| \mathcal{T}_\varepsilon v^\varepsilon_0 - V^{\mathrm{ex}}_0 \|_{\mathbb{H}_{\mathbb{R}^d}} \le \varepsilon^{1/4} c. \end{aligned} \tag{2.3.63}$$

The initial values considered in Proposition 2.3.17, below, satisfy also assumption (2.3.52) in Theorem 2.3.13 for the improved estimate on the interior Ω_ϱ. So, we derive a quantification of the choice of initial values for the homogenization result with improved time-regularity, cf. Proposition 2.2.7.

Before, we briefly comment on the satisfiability of assumption (2.3.4) in Main Theorem IIIb for solutions $(u^\varepsilon, v^\varepsilon)$ without improved time-regularity. For given $u_0 \in X$, the constant sequence $u_0^\varepsilon := u_0$ is admissible. Now, let $V_0 \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ be given. The choice $v_0^\varepsilon := \widetilde{\mathcal{G}}_\varepsilon^1 V_0 \in X_\varepsilon$ is canonical, since Lemma 2.3.4 and Proposition 2.3.8 immediately imply $\| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}} \|_{\mathbb{H}_{\mathbb{R}^d}} \leq \sqrt{\varepsilon} C$. If V_0 is additionally continuous, we can set as well $v_0^\varepsilon(x) := V_0(x, x/\varepsilon)$, cf. Lemma 2.3.5. Due to the "true" two-scale character of the slowly diffusing species v^ε, the constant choice – as for u_0^ε – is not meaningful.

Let us focus on initial values implying improved-time regularity as in (2.3.63). As before, such quantitative estimates are established in the case of classical diffusion (see also Table 2.1 in Subsection 2.3.9 for an overview).

Proposition 2.3.15 ([Gri04, Prop. 4.3])**.** *Let the coefficients be of the form $D^\varepsilon(x) = \mathbb{D}(x/\varepsilon)$ with $\mathbb{D} \in \mathcal{M}(\mathcal{Y})$. For right-hand sides $f \in H$, we choose $u_0^\varepsilon, u_0 \in X$ as the unique solutions of the elliptic problems*

$$u^\varepsilon - \operatorname{div}(D^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega \qquad \text{and} \qquad u - \operatorname{div}(D_{\mathrm{eff}} \nabla u) = f \quad \text{in } \Omega. \tag{2.3.64}$$

If the higher regularity $u \in \mathrm{H}^2(\Omega)$ holds, then there exists a constant $C \geq$ such that

$$\|u_0^\varepsilon - u_0\|_H + \left\| \nabla u + \textstyle\sum_{j=1}^d \mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_j} \right) \cdot \nabla_y z_j(\frac{\cdot}{\varepsilon}) - \nabla u^\varepsilon \right\|_H \leq \varepsilon^{1/2} C.$$

Using the decomposition $U(x,y) = \sum_{j=1}^d \frac{\partial u}{\partial x_j}(x) \cdot z_j(y)$ of the corrector, the gradient estimate in H (via $\mathcal{Q}_\varepsilon$) is equivalent to the corresponding one in $\mathbb{H}$ (via $\mathcal{T}_\varepsilon$). Indeed Lemma 2.3.4 and 2.3.7 imply

$$\left\| \nabla u + \textstyle\sum_{j=1}^d \mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_j} \right) \cdot \nabla_y z_j(\frac{\cdot}{\varepsilon}) - \nabla u^\varepsilon \right\|_H = \| \nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \|_{\mathbb{H}} + O(\varepsilon + \sigma\sqrt{\varepsilon}).$$

The exact periodicity of the coefficients D^ε in Proposition 2.3.15 yields corrector functions z_{ij} which are independent of $x \in \Omega$. With this, the corrector $U(x,y)$ has the above product form and the scale-splitting operator $\mathcal{Q}_\varepsilon$ is directly applicable (cf. Step 2 of the proof to Proposition 2.3.8). It is not immediately clear, whether Proposition 2.3.15 can be generalized to not exactly periodic coefficients $\mathbb{D}(x, x/\varepsilon)$.

Based on our preparatory estimates for the error terms, we obtain the following two propositions so that the assumptions (2.3.2.A4), cf. (2.3.63), and (2.3.52) of Theorem 2.3.1 and 2.3.13, respectively, are satisfied. To this, let the assumptions (2.3.2.A0)–(2.3.2.A3) hold, in particular, $x \mapsto \mathbb{D}_i(0, x, y)$ is Lipschitz continuous.

Proposition 2.3.16. *For right-hand sides $f \in H$, we choose $u_0^\varepsilon, u_0 \in X$ as the unique solutions of the elliptic problems*

$$u_0^\varepsilon - \operatorname{div}(D_1^\varepsilon(0) \nabla u_0^\varepsilon) = f \quad \text{in } \Omega \quad \text{and} \quad u_0 - \operatorname{div}(D_{\mathrm{eff}}(0) \nabla u_0) = f \quad \text{in } \Omega.$$

If the higher regularity $u_0 \in \mathrm{H}^2(\Omega)$ is satisfied, then there exists $c \geq 0$ such that

$$\|u_0^\varepsilon - u_0\|_H + \| \mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]^{\mathrm{ex}} \|_{\mathbb{H}_{\mathbb{R}^d}} \leq \varepsilon^{1/4} c; \tag{2.3.65}$$

Proposition 2.3.17. *For right-hand sides $G \in \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y}))$, we choose $v_0^\varepsilon \in X_\varepsilon$ and $V_0 \in \mathbb{X}$ as the unique solutions of the elliptic problems*

$$\begin{aligned} v_0^\varepsilon - \mathrm{div}(\varepsilon^2 D_2^\varepsilon(0)\nabla v_0^\varepsilon) &= \mathcal{F}_\varepsilon G \qquad && \text{in } \Omega, \\ V_0 - \mathrm{div}_y(\mathbb{D}_2(0)\nabla_y V_0) &= G && \text{in } \Omega \times \mathcal{Y}. \end{aligned} \tag{2.3.66}$$

Then, we have the additional regularity $V_0 \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ and it holds

$$\text{(a)} \quad \exists\, c \geq 0 : \quad \| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}} \|_{\mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1(Y))} \leq \varepsilon^{1/4} c; \tag{2.3.67}$$

$$\text{(b)} \quad \forall\, \varrho > 0\ \exists\, c_\varrho \geq 0\ \forall\, \varepsilon \leq \varrho/(2\sqrt{d}) : \quad \| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0 \|_{\mathrm{L}^2(\Omega_\varrho; \mathrm{H}^1(Y))} \leq \varepsilon^{1/2} c_\varrho. \tag{2.3.68}$$

The derivation of the estimates (2.3.65) and (2.3.67)–(2.3.68), is done by the same method. As before, we reformulate the original respective the effective problem by creating error terms for the folding mismatch, the periodicity defect, and the approximations of the given data. Then, taking the difference of the reformulated problems and controlling the error terms by the preparatory estimates yields the desired convergence rate $\varepsilon^{1/4}$ on Ω respective $\varepsilon^{1/2}$ on Ω_ϱ for the solutions. We only carry out the proof of Proposition 2.3.17 (slow diffusion) and Proposition 2.3.16 (classical diffusion) follows analogously by applying the corresponding preparatory estimates with $\gamma = 0$ instead of $\gamma = 1$. Notice that higher regularity is assumed for the limit u_0, while for V_0 it is immediate by the higher regularity of the given data.

Proof of Proposition 2.3.17. *Step 1: Higher regularity of the limit V_0.* To see that $V_0 \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$, we estimate the difference quotient $\frac{1}{h}(V_0(x + he_j, y) - V_0(x, y))$ for $h > 0$. Let $\{e_j\}_j$ denote the canonical orthonormal basis in $\mathbb{R}^d$ and $\widetilde{\Omega}$ a compact subset of Ω. We define for almost every $x \in \widetilde{\Omega}$ and $h < \mathrm{dist}(\widetilde{\Omega}, \partial\Omega)$ the following abbreviations $\tilde{x} := x + he_j$, $V := V_0(x, \cdot)$, $\widetilde{V} := V_0(\tilde{x}, \cdot)$, $\mathbb{D} := \mathbb{D}_2(0, x, \cdot)$, $\widetilde{\mathbb{D}} := \mathbb{D}_2(0, \tilde{x}, \cdot)$, $G := G(x, \cdot)$, and $\widetilde{G} := G(\tilde{x}, \cdot)$. Clearly, the functions $V, \widetilde{V} \in \mathbb{X}$ solve the elliptic problems

$$V - \mathrm{div}_y(\mathbb{D}\nabla_y V) = G \quad \text{and} \quad \widetilde{V} - \mathrm{div}_y(\widetilde{\mathbb{D}}\nabla_y \widetilde{V}) = \widetilde{G}$$

on $\mathcal{Y}$. Taking their difference, integrating over $\widetilde{\Omega} \times \mathcal{Y}$ and testing with $\Phi \in \mathbb{X}$, as well as dividing by h and setting $W_h := \frac{1}{h}(\widetilde{V} - V)$, yields

$$\int_{\widetilde{\Omega} \times \mathcal{Y}} W_h \cdot \Phi + \widetilde{\mathbb{D}}\nabla_y W_h : \nabla_y \Phi \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{h} \int_{\widetilde{\Omega} \times \mathcal{Y}} (\mathbb{D} - \widetilde{\mathbb{D}})\nabla_y V : \nabla_y \Phi + (\widetilde{G} - G) \cdot \Phi \,\mathrm{d}x\,\mathrm{d}y.$$

The Lipschitz continuity of $\mathbb{D}_2$ w.r.t. $x \in \Omega$ gives $\|\mathbb{D} - \widetilde{\mathbb{D}}\|_{\mathrm{L}^\infty(\widetilde{\Omega} \times \mathcal{Y})} \leq hD_\infty$, where $D_\infty := \|\partial_{x_j}\mathbb{D}\|_{\mathrm{L}^\infty(\Omega \times \mathcal{Y})}$. Choosing $\Phi = W_h$, yields the estimate

$$\min\{1, \alpha\} \|W_h\|^2_{\mathrm{L}^2(\widetilde{\Omega}; \mathrm{H}^1(\mathcal{Y}))} \leq \Big(D_\infty \|\nabla_y V\|_{\mathrm{L}^2(\widetilde{\Omega} \times \mathcal{Y})} + \|\tfrac{1}{h}(\widetilde{G} - G)\|_{\mathrm{L}^2(\widetilde{\Omega} \times \mathcal{Y})} \Big) \|W_h\|_{\mathrm{L}^2(\widetilde{\Omega}; \mathrm{H}^1(\mathcal{Y}))}.$$

With $\|\nabla_y V\|_{\mathrm{L}^2(\widetilde{\Omega} \times \mathcal{Y})} \leq \|\nabla_y V\|_{\mathbb{H}}$ and $\|\frac{1}{h}(\widetilde{G} - G)\|_{\mathrm{L}^2(\widetilde{\Omega} \times \mathcal{Y})} \leq \|\partial_{x_j} G\|_{\mathbb{H}}$ according to [GiT01, Lem. 7.23], we obtain the uniform bound $\|W_h\|_{\mathrm{L}^2(\widetilde{\Omega}; \mathrm{H}^1(\mathcal{Y}))} \leq K$, where $K \geq 0$ is independent of $h > 0$ and $\widetilde{\Omega}$. According to [GiT01, Lem. 7.24], this implies the existence of the weak derivative $\partial_{x_j} V_0$ with $\|\partial_{x_j} V_0\|_{\mathbb{X}} \leq C$, where $C \geq 0$ depends on α, $\|V_0\|_{\mathbb{X}}$, $\|\partial_{x_j} G\|_{\mathbb{H}}$, and $\|\partial_{x_j} \mathbb{D}_2(0)\|_{\mathrm{L}^\infty(\Omega \times \mathcal{Y})}$.

Step 2: Error Estimates. We follow our approach to the system of coupled parabolic equations and adapt it to the elliptic problems in (2.3.66). As before, we estimate the difference

$$W_0^\varepsilon := \mathcal{T}_\varepsilon\, v_0^\varepsilon - V_0^{\mathrm{ex}} \quad \text{in } \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1(Y))$$

and set $G^\varepsilon := \mathcal{F}_\varepsilon\, G^{\mathrm{ex}}$ as well as $D_2^\varepsilon := D_2^\varepsilon(0)$ respective $\mathbb{D}_2 := \mathbb{D}_2(0)$.

Step 2a. We prove (2.3.67) first. We test the weak formulation of $(2.3.66)_1$ with $v_0^\varepsilon - \widetilde{\mathcal{G}}_\varepsilon^1\, V_0 \in X_\varepsilon$ and use $\mathcal{F}_\varepsilon\, \mathcal{T}_\varepsilon = \mathrm{id}|_H$ so that

$$\int_\Omega v_0^\varepsilon \cdot \mathcal{F}_\varepsilon\, W_0^\varepsilon + D_2^\varepsilon \varepsilon \nabla v_0^\varepsilon : \mathcal{F}_\varepsilon(\nabla_y W_0^\varepsilon)\,\mathrm{d}x = \int_\Omega G^\varepsilon \cdot \mathcal{F}_\varepsilon\, W_0^\varepsilon\,\mathrm{d}x + \Delta_1^{v_0^\varepsilon}, \tag{2.3.69}$$

$$\text{where } \Delta_1^{v_0^\varepsilon} := \int_\Omega (G^\varepsilon - v_0^\varepsilon) \cdot (\mathcal{F}_\varepsilon\, V_0^{\mathrm{ex}} - \widetilde{\mathcal{G}}_\varepsilon^1\, V_0) + D_2^\varepsilon \varepsilon \nabla v_0^\varepsilon : [\mathcal{F}_\varepsilon(\nabla_y V_0)^{\mathrm{ex}} - \varepsilon\nabla(\widetilde{\mathcal{G}}_\varepsilon^1\, V_0)]\,\mathrm{d}x.$$

Exploiting the duality $\mathcal{T}_\varepsilon = \mathcal{F}_\varepsilon{}'$ and $\mathcal{T}_\varepsilon(\varepsilon\nabla v) = \nabla_y(\mathcal{T}_\varepsilon\, v)$ gives

$$\int_{\mathbb{R}^d\times\mathcal{Y}} \mathcal{T}_\varepsilon\, v_0^\varepsilon \cdot W_0^\varepsilon + \mathcal{T}_\varepsilon\, D_2^\varepsilon \nabla_y(\mathcal{T}_\varepsilon\, v_0^\varepsilon) : \nabla_y W_0^\varepsilon\,\mathrm{d}x = \int_{\Omega\times\mathcal{Y}} \mathcal{T}_\varepsilon\, G^\varepsilon \cdot W_0^\varepsilon\,\mathrm{d}x + \Delta_1^{v_0^\varepsilon}. \tag{2.3.70}$$

We test the weak formulation of $(2.3.66)_2$ with V_0, extend with 0 to $\mathbb{R}^d\backslash\Omega$, and insert the terms $\pm\,\mathcal{T}_\varepsilon\, v_0^\varepsilon$ resp. $\pm\nabla_y(\mathcal{T}_\varepsilon\, v_0^\varepsilon)$, i.e.

$$\int_{\mathbb{R}^d\times\mathcal{Y}} V_0^{\mathrm{ex}} \cdot W_0^\varepsilon + \mathbb{D}_2^{\mathrm{ex}} \nabla_y V_0^{\mathrm{ex}} : \nabla_y W_0^\varepsilon\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d\times\mathcal{Y}} G^{\mathrm{ex}} \cdot W_0^\varepsilon\,\mathrm{d}x + \Delta_2^{v_0^\varepsilon}, \tag{2.3.71}$$

$$\text{where } \Delta_2^{v_0^\varepsilon} := \int_{\mathbb{R}^d\times\mathcal{Y}} (V_0^{\mathrm{ex}} - G^{\mathrm{ex}}) \cdot \mathcal{T}_\varepsilon\, v_0^\varepsilon - \mathbb{D}_2^{\mathrm{ex}} \nabla_y V_0^{\mathrm{ex}} : \nabla_y(\mathcal{T}_\varepsilon\, v_0^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y.$$

Taking the difference of (2.3.70) and (2.3.71) yields

$$\int_{\mathbb{R}^d\times\mathcal{Y}} W_0^\varepsilon \cdot W_0^\varepsilon + \mathcal{T}_\varepsilon\, D_2^\varepsilon \nabla_y W_0^\varepsilon : \nabla_y W_0^\varepsilon\,\mathrm{d}x\,\mathrm{d}y = \Delta^{v_0^\varepsilon}, \tag{2.3.72}$$

where $\Delta^{v_0^\varepsilon} = \sum_{i=1}^4 \Delta_i^{v_0^\varepsilon}$ with

$$\begin{aligned}
\Delta_3^{v_0^\varepsilon} &:= \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon\, D_2^\varepsilon - \mathbb{D}_2^{\mathrm{ex}}) \nabla_y V_0^{\mathrm{ex}} : \nabla_y W_0^\varepsilon\,\mathrm{d}x\,\mathrm{d}y,\\
\Delta_4^{v_0^\varepsilon} &:= \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon\, G^\varepsilon - G^{\mathrm{ex}}) \cdot W_0^\varepsilon\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The error terms $\Delta_i^{v_0^\varepsilon}$ resemble those in Subsection 2.1.5 (see page 38) and we derive quantitative estimates as in Subsection 2.3.4. Using the uniform boundedness of the given data and of v_0^ε in X_ε, we obtain with Lemma 2.3.8 for the folding mismatch

$$|\Delta_1^{v_0^\varepsilon}| \le C\Big\{\|\,\mathcal{F}_\varepsilon\, V_0^{\mathrm{ex}} - \widetilde{\mathcal{G}}_\varepsilon^1\, V_0\|_H + \|\,\mathcal{F}_\varepsilon(\nabla_y V_0)^{\mathrm{ex}} - \varepsilon\nabla(\widetilde{\mathcal{G}}_\varepsilon^1\, V_0)\|_H\Big\} \le (\varepsilon + \sqrt{\varepsilon})C.$$

To estimate the periodicity defect error, we introduce $\Psi^\varepsilon \in \mathbb{X}$ as in Lemma 2.3.9 and exploit the additional $\mathrm{H}^1(\Omega)$-regularity of G and V_0 resp. $\partial_{x_j}\mathbb{D}_2 \in \mathrm{L}^\infty(\Omega\times\mathcal{Y})$ so that

$$|\Delta_2^{v_0^\varepsilon}| \le C\|\,\mathcal{T}_\varepsilon\, v_0^\varepsilon - \Psi^\varepsilon\|_{\mathrm{H}^1(Y;X^*)} \le (\varepsilon + \sqrt{\varepsilon})C.$$

Applying Lemma 2.3.4 and 2.3.5 to the approximation errors yields $|\Delta_3^{v_0^\varepsilon}| + |\Delta_4^{v_0^\varepsilon}| \leq (\varepsilon + \sqrt{\varepsilon})C$. Therefore, estimating the left-hand side in (2.3.72) using the uniform ellipticity, we arrive at

$$\| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0^{\mathrm{ex}} \|^2_{\mathrm{L}^2(\mathbb{R}^d;\mathrm{H}^1(Y))} \leq (\varepsilon + \sqrt{\varepsilon})C. \tag{2.3.73}$$

Step 2b. We now turn to estimate (2.3.68). For fixed $\varrho > 0$, let $\vartheta_\varrho \in C_c^\infty(\Omega)$ be a nonnegative cut-off function such that

$$\vartheta_\varrho \equiv 1 \text{ on } \Omega_\varrho \quad \text{and} \quad \vartheta_\varrho \equiv 0 \text{ on } \Omega \backslash \Omega_{\varrho/2}.$$

We choose the test functions $\vartheta_\varrho^2 v_0^\varepsilon - \mathcal{G}_\varepsilon^1(\vartheta_\varrho^2 V_0) \in X_\varepsilon$ and $\vartheta_\varrho^2 V_0 \in \mathbb{X}$ in (2.3.69) and (2.3.71), respectively. Repeating the argumentations of Step 2a as well as using the reformulations in (2.3.60), we obtain the periodicity defect

$$\Delta_{\varrho,2}^{v_0^\varepsilon} := \int_{\Omega\times\mathcal{Y}} \vartheta_\varrho (V_0 - G) \cdot \mathcal{T}_\varepsilon(\vartheta_\varrho v_0^\varepsilon) - \mathbb{D}_2 \nabla_y (\vartheta_\varrho V_0) : \nabla_y([\mathcal{T}_\varepsilon(\vartheta_\varrho v_0^\varepsilon]) \,\mathrm{d}x\,\mathrm{d}y$$

which is estimated by εC_ϱ with Lemma 2.3.10 as in (2.3.62). For the estimation of the remaining error terms, we use the estimates in (2.3.61) and restrict ourselves to Ω_ε^- for all $\varepsilon < \varrho/(2\sqrt{d})$ so that $\sigma = 0$. Then, the desired estimate (2.3.68) follows as in Step 2a. □

2.3.8 On the improved regularity of the effective solutions

We evaluate the satisfiability of assumption (2.3.2.A3) on the improved regularity of the effective solution (u, V) of (2.0.2.P$_0$), namely

$$\begin{aligned} &u \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)), \\ &V \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))) \cap \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))). \end{aligned} \tag{2.3.74}$$

First, we consider the classical diffusion case for vector-valued functions $u : \Omega \to \mathbb{R}^m$ and recall the notation $X = [\mathrm{H}^1(\Omega)]^m$ and $H = [\mathrm{L}^2(\Omega)]^m$ for arbitrary $m \in \mathbb{N}$. For linear elliptic systems, the following regularity result holds.

Theorem 2.3.18 ([CDN10, Thm. 3.4.1]). *Let Ω be of class C^2. We assume for the elliptic diffusion coefficients $D \in C^1(\overline{\Omega};\mathbb{R}^{(m\times d)\times(m\times d)})$ and for the right-hand side $f \in H$. Let $u \in X$ denote the solution of the elliptic system*

$$u - \mathrm{div}(D\nabla u) = f \quad \textit{in } \Omega \quad \textit{with} \quad (D\nabla u)\cdot \nu = 0 \quad \textit{on } \partial\Omega.$$

Then, u belongs to the space $\mathrm{H}^2(\Omega)$ and satisfies the estimate

$$\|u\|_{\mathrm{H}^2(\Omega)} \leq C\left(\|f\|_H + \|u\|_X\right).$$

Let the smoothness assumptions in Theorem 2.3.18 for Ω and $D_{\mathrm{eff}}(t)$ hold for all $t \in [0,T]$ throughout the remainder of this subsection. With this, the limit solution u_0 of the system (2.3.64)$_2$ in Proposition 2.3.16 satisfies $u_0 \in \mathrm{H}^2(\Omega)$.

We briefly outline how the regularity for linear elliptic systems can be generalized to systems of semilinear parabolic equations such as the effective equations $(2.0.2.\mathrm{P}_0)_1$. Therefore, we consider in a first step improved time-regularity for the solutions $u : [0,T] \times \Omega \to \mathbb{R}^{m_1}$ and $V : [0,T] \times \Omega \times \mathcal{Y} \to \mathbb{R}^{m_2}$ of the whole system $(2.0.2.\mathrm{P}_0)$, namely

$$\begin{aligned} u_t &= \operatorname{div}(D_{\mathrm{eff}} \nabla u) + F_{\mathrm{eff}}(u,V) \quad \text{in } [0,T] \times \Omega, \\ V_t &= \operatorname{div}_y(\mathbb{D}_2 \nabla_y V) + \mathbb{F}_2(u,V) \quad \text{in } [0,T] \times \Omega \times \mathcal{Y}. \end{aligned} \tag{2.3.75}$$

Let the initial values $u_0 \in H$ and $V_0 \in \mathbb{H}$ satisfy the additional assumption

$$\operatorname{div}(D_{\mathrm{eff}}(0)\nabla u_0) \in H \quad \text{and} \quad \operatorname{div}_y(\mathbb{D}_2(0)\nabla_y V_0) \in \mathbb{H},$$

which imply $u \in W_{\mathrm{imp}}(0,T;X)$ and $V \in W_{\mathrm{imp}}(0,T;\mathbb{X})$ by Proposition 1.1.3.

In the second step, we derive the higher x-regularity for the solution (u,V). For the slow diffusion limit V, the higher x-regularity follows directly from the higher regularity of the given data and we obtain $V \in C([0,T];\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))) \cap C^1([0,T];\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y})))$.

For the classical solution u, the higher regularity is not so straight forward. Let $0 = t_0^N < t_1^N < \ldots < T_N^N = T$ denote an equidistant partition of the time interval $[0,T]$ with step size $\tau_N = T/N$ for $N \in \mathbb{N}$. Approximating $u(t_n^N)$ and $V(t_n^N)$ with u_n^N and V_n^N, respectively, and using a semi-implicit time-discretization scheme for $(2.3.75)_1$ gives

$$\tfrac{1}{\tau_N}\left(u_{n+1}^N - u_n^N\right) = \operatorname{div}(D_{\mathrm{eff}}(t_{n+1}^N)\nabla u_{n+1}^N) + F_{\mathrm{eff}}(t_n^N, u_n^N, V_n^N). \tag{2.3.76}$$

Solving this linear elliptic system for every $n = 0, \ldots, N-1$ and applying Theorem 2.3.18 yields $u_{n+1}^N \in \mathrm{H}^2(\Omega)$. Moreover, the growth condition (1.1.8) for $\mathbb{F}_1$ and the improved time-regularity $u_t \in C^1([0,T];H)$ imply

$$\|u_{n+1}^N\|_{\mathrm{H}^2(\Omega)} \leq C\left(\|F_{\mathrm{eff}}(t_n^N, u_n^N, V_n^N)\|_H + \|\tfrac{1}{\tau_N}(u_{n+1}^N - u_n^N)\|_H + \|u_{n+1}^N\|_X\right) \leq C(C_b).$$

Here, C_b is the uniform bound from (2.1.13), independent of N. Since $\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ embeds continuously into $\mathrm{H}^1(\Omega \times \mathcal{Y})$, we obtain the uniform the a priori bound

$$\sup_{N\in\mathbb{N}}\{\|u_n^N\|_X + \|V_n^N\|_{\mathrm{H}^1(\Omega\times\mathcal{Y})}\} < \infty$$

so that we can pass to the limit $N \to \infty$ with the nonlinear term $F_{\mathrm{eff}}(t_n^N, u_n^N, V_n^N)$ exploiting the compact embeddings $X \subset H$ and $\mathrm{H}^1(\Omega \times \mathcal{Y}) \subset \mathbb{H}$. Proceeding as in e.g. [Emm04, Sec. 8.3], we can pass to the limit $N \to \infty$ in the time-discrete problem (2.3.76) and obtain $u \in W_{\mathrm{imp}}(0,T;X) \cap C([0,T];\mathrm{H}^2(\Omega))$ as solution of the time-continuous problem $(2.3.75)_1$. Overall, we obtain (2.3.74).

2.3.9 Comparison with related results

We conclude Section 2.3 with a subsumption of our results and compare them with related results in the literature. Let the effective solutions be of higher spatial regularity, i.e. $u \in \mathrm{H}^2(\Omega)$ for classical diffusion and $V \in \mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ for slow diffusion. We collect the following results.

In [Gri04, Gri05], the author considers linear elliptic equations with exactly periodic coefficients $\mathbb{D}(x/\varepsilon)$ combined with Dirichlet or Neumann boundary values. Depending on

norm	elliptic	parabolic	classical and slow diffusion
$\Vert u^\varepsilon - u\Vert_H +$ $\Vert \nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\Vert_{\mathbb{H}}$	$\varepsilon^{1/2}$ (Prop. 2.3.15)	$\varepsilon^{1/2}$ (Rem. 2.3.19)	-
$\varepsilon^{-1/4}\Vert u^\varepsilon - u\Vert_{\mathrm{L}^2(0,T;H)} +$ $\Vert \nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\Vert_{\mathrm{L}^2(0,T;\mathbb{H})}$	-	$\varepsilon^{1/4}$ [FMP12]	-
$\Vert v^\varepsilon - [V]^\varepsilon\Vert_{\mathrm{L}^\infty(0,T;H)} +$ $\Vert \varepsilon\nabla v^\varepsilon - \nabla_y [V]^\varepsilon\Vert_{\mathrm{L}^2(0,T;H)}$	-	-	$\varepsilon^{1/2}$ [Eck05]
$\Vert v^\varepsilon - [V]^\varepsilon\Vert_{\mathrm{L}^\infty(0,T;H)} +$ $\Vert \varepsilon\nabla v^\varepsilon - \nabla_y [V]^\varepsilon\Vert_{\mathrm{L}^\infty(0,T;H)}$	-	-	$\varepsilon^{1/2}$ [Muv13]
$\Vert \mathcal{T}_\varepsilon v^\varepsilon - V\Vert_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))}$	$\varepsilon^{1/4}$ (Prop. 2.3.17)	-	-
$\Vert \mathcal{T}_\varepsilon v^\varepsilon - V\Vert_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega;\mathrm{H}^1(Y)))}$	-	-	$\varepsilon^{1/4}$ (Thm. 2.3.1)
interior estimates			
$\Vert u^\varepsilon - u\Vert_H +$ $\Vert \nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\Vert_{\mathrm{L}^2(\Omega_\varepsilon\times\mathcal{Y})}$	ε [Gri05]	-	-
$\Vert \mathcal{T}_\varepsilon v^\varepsilon - V\Vert_{\mathrm{L}^2(\Omega_\varrho;\mathrm{H}^1(Y))}$	$\varepsilon^{1/2}$ (Prop. 2.3.17)	-	-
slow diffusion only $\Vert \mathcal{T}_\varepsilon v^\varepsilon - V\Vert_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_\varrho;\mathrm{H}^1(Y)))}$	-	$\varepsilon^{1/2}$ (Thm. 2.3.13)	-

Table 2.1: Comparison of convergence rates. Here, Ω_ε and Ω_ϱ denote subsets of Ω with $\mathrm{dist}(\Omega_\varepsilon, \partial\Omega) \sim \varepsilon$ and $\mathrm{dist}(\Omega_\varrho, \partial\Omega) \sim \varrho$ for fixed $\varrho > 0$, respectively. Recall that $[V]^\varepsilon(x) = V(x, x/\varepsilon)$.

the regularity of the boundary $\partial\Omega$, the convergence rates $\varepsilon^{1/2}$ and ε are derived. These error estimates are extendable to solutions of semilinear parabolic equations and systems, see Remark 2.3.19. We refer to [OnV07] and references therein for improved estimates also based on the periodic unfolding method and more regularity of the limit solution.

In [FMP12], a system of reaction-diffusion equations is considered on a cubical domain $\Omega \subset \mathbb{R}^3$ with exactly periodic, porous microstructure. The system does not include slowly diffusing species v^ε, but rather nonlinear boundary conditions at the surface of the pores. For the classically diffusing species u^ε the convergence rate $\varepsilon^{1/4}$ is rigorously proved by the method of periodic unfolding. We emphasize that the non-gradient estimate is better, namely, $\|u^\varepsilon - u\|_{\mathrm{L}^2(0,T;H)} \le \varepsilon^{1/2}$. These error estimates are comparable with the one in Main Theorem IIIa. Due to the nonlinear coupling of u^ε and v^ε, it is not to expect that the convergence rate for $\|u^\varepsilon - u\|_{\mathrm{L}^2(0,T;H)}$ can be improved to $\varepsilon^{1/2}$.

In [Eck05, Muv13], systems of nonlinearly coupled reaction-diffusion equations involving diffusion length scales of order $O(1)$ and $O(\varepsilon)$ are considered in a heterogeneous setting. Whereas in [Eck05] the coefficient functions are of the form $\mathbb{D}(x, x/\varepsilon)$, in [Muv13], the heterogeneities in the domain $\Omega \subset \mathbb{R}^2$ are only locally-periodic. In both cases, the approach of formal asymptotic expansion is used and, then, convergence rates are proved under the assumption of significantly more spatial regularity for the effective solutions; in particular differentiability w.r.t. $y \in \mathcal{Y}$ is required. Assuming $V \in W^{1,\infty}(\Omega; C^1(\mathcal{Y}))$ in the case of slow diffusion, we obtain with Lemma 2.3.5

$$\|v^\varepsilon - [V]^\varepsilon\|_H + \|\varepsilon\nabla v^\varepsilon - \nabla_y [V]^\varepsilon\|_H = \|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{X}} + O(\varepsilon + \sigma\sqrt{\varepsilon}).$$

For slow diffusion only, our method reproduces in the interior $\Omega_\varrho \subsetneqq \Omega$ the convergence rates in [Eck05, Thm. 4.5] and [Muv13, Thm. 3.1] without assuming any continuity for the effective solutions. So far, it is not clear how the better rate $\varepsilon^{1/2}$ can be proved with our methods for systems coupling u^ε and v^ε. Moreover, we cannot take $\varrho \sim \varepsilon$ in Theorem 2.3.13, since the constant C_ϱ depends on the norm of the cut-off function ϑ_ϱ via $\|\vartheta_\varrho\|_{W^{1,\infty}(\Omega)} \lesssim \frac{1}{\varrho}$.

Slow diffusion versus classical diffusion. Even for linear elliptic equations, the convergence rate for u^ε in Proposition 2.3.15 is twice as good as the rate for v^ε in Proposition 2.3.17, namely $\varepsilon^{1/2}$ and ε compared to $\varepsilon^{1/4}$ and $\varepsilon^{1/2}$ on Ω and Ω_ϱ, respectively (cf. Table 2.1 “elliptic”). Note that, we have a priori $u^\varepsilon \to u$ *strongly* and $\nabla u^\varepsilon \xrightharpoonup{2\mathrm{w}} \nabla u + \nabla_y U$ *weakly* versus $v^\varepsilon \xrightharpoonup{2\mathrm{w}} V$ *and* $\varepsilon \nabla v^\varepsilon \xrightharpoonup{2\mathrm{w}} \nabla_y V$ *weakly.* Therefore, one should qualitatively and quantitatively compare the convergence of v^ε with ∇u^ε – and not with u^ε. Nevertheless, there remains to explain a gap between the two results. In the proof of Proposition 2.3.17, we obtain an estimate of the form, cf. (2.3.73),

$$\|\mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))} = \Delta^{v^\varepsilon} \le (\varepsilon + \sigma\sqrt{\varepsilon})C,$$

which immediately implies the convergence rate $\varepsilon^{1/2}$ respective $\varepsilon^{1/4}$ with $\sigma \in \{0, 1\}$ depending on the boundary properties. Here, the constant C depends among others on the relevant norms of v^ε and V. It would be desirable that the right-hand side also depends on the difference $\mathcal{T}_\varepsilon v^\varepsilon - V$, namely (see also estimate (2.3.78) in Remark 2.3.19 below)

$$\|\mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))} \le (\varepsilon + \sigma\sqrt{\varepsilon})C\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))}.$$

However, for the folding mismatch $\Delta_1^{v^\varepsilon}$ and the periodicity defect $\Delta_2^{v^\varepsilon}$ such an improved estimate is not obvious. It is unclear whether this drawback is due to the methods of Chapter 2 and the way the error terms are derived or this drawback is inherent to the problem of slow diffusion. Indeed, for classically diffusing species the periodicity defect only arises for the gradient $\mathcal{T}_\varepsilon(\nabla u^\varepsilon)$ and not for the function u^ε itself. Moreover, the folding mismatch between $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^0$ does not occur in [Gri04, Gri05], since the function $U_\varepsilon(x, x/\varepsilon) = \sum_{j=1}^d \mathcal{Q}_\varepsilon\left(\frac{\partial u}{\partial x_j}\right)(x) \cdot z_j(x/\varepsilon)$ possesses sufficient regularity to be an admissible test function. In contrast, the effective solution V is in general not a product (i.e. $V(x,y) \neq w(x)z(y)$) and, hence, the introduction of the gradient folding operator $\mathcal{G}_\varepsilon^1$ seems inevitable. The same holds for the corrector U, if $\mathbb{D}(x, x/\varepsilon)$ is not exactly periodic and the corrector functions $z_j(x,y)$ depend additionally on $x \in \Omega$.

Remark 2.3.19. *We study the convergence rate for linear parabolic problems in the absence of slow diffusion. In [Gri04, Prop. 4.3] the elliptic equation* $\operatorname{div}(D^\varepsilon \nabla u^\varepsilon) = f$ *is considered with* $f \in H$ *and* $D^\varepsilon(x) := \mathbb{D}(x/\varepsilon)$ *with* $\mathbb{D} \in \mathrm{L}^\infty(\mathcal{Y})$*. If the solution of the effective equation* $\operatorname{div}(D_{\mathrm{eff}} \nabla u) = f$ *satisfies* $u \in \mathrm{H}^2(\Omega)$*, then it holds*

$$\|u^\varepsilon - u\|_H + \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]\|_{\mathbb{H}} \leq \sqrt{\varepsilon} C. \tag{2.3.77}$$

We briefly recall the arguments of the proof, where we set

$$w^\varepsilon(x) := \sum_{j=1}^d \mathcal{Q}_\varepsilon\left(\frac{\partial u}{\partial x_j}\right)(x) \cdot z_j(\tfrac{x}{\varepsilon}) \quad \text{and} \quad \nabla_y w^\varepsilon(x) := \sum_{j=1}^d \mathcal{Q}_\varepsilon\left(\frac{\partial u}{\partial x_j}\right)(x) \cdot \nabla_y z_j(\tfrac{x}{\varepsilon}).$$

Testing the effective equation with $\varphi \in X$*, we obtain after numerous estimates and reformulations*

$$\left| \int_\Omega f \cdot \varphi - D^\varepsilon[\nabla u + \nabla_y w^\varepsilon] : \nabla \varphi \,\mathrm{d}x \right| \leq \sqrt{\varepsilon} C \|\varphi\|_X.$$

For the u^ε*-equation, we immediately obtain* $\int_\Omega f \cdot \varphi - D^\varepsilon \nabla u^\varepsilon : \nabla \varphi \,\mathrm{d}x = 0$*. We insert the test function* $\varphi^\varepsilon = u + \varepsilon w^\varepsilon - u^\varepsilon$ *and note* $\|\varepsilon w^\varepsilon\|_X \leq \sqrt{\varepsilon} C$*. Taking the difference of the two equations and using the uniform ellipticity as well as some Poincaré-type inequality yields*

$$(\|u - u^\varepsilon\|_H + \|u + \nabla_y w^\varepsilon - u^\varepsilon\|_H)^2 \leq \sqrt{\varepsilon} C \left(\|u - u^\varepsilon\|_H + \|u + \nabla_y w^\varepsilon - u^\varepsilon\|_H\right), \tag{2.3.78}$$

which implies (2.3.77)*.*

Now, let us consider the associated parabolic equation $u_t^\varepsilon + \operatorname{div}([\mathbb{D}]^\varepsilon \nabla u^\varepsilon) = f$*. Repeating the previous calculations yields*

$$\left| \int_\Omega (u_t - f) \cdot \varphi - D^\varepsilon[\nabla u + \nabla_y w^\varepsilon] : \nabla \varphi \,\mathrm{d}x \right| \leq \sqrt{\varepsilon} C \|\varphi\|_X$$

and $\int_\Omega (u_t^\varepsilon - f) \cdot \varphi - [\mathbb{D}]^\varepsilon \nabla u^\varepsilon : \nabla \varphi \,\mathrm{d}x = 0$ *almost everywhere in* $[0,T]$*. Testing again with* $\varphi^\varepsilon = u + \varepsilon w^\varepsilon - u^\varepsilon$*, subtracting both equations and using Young's inequality with* $\alpha > 0$ *being the ellipticity constant gives*

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 &\leq -\int_\Omega D^\varepsilon[\nabla u + \nabla_y w^\varepsilon - \nabla u^\varepsilon] : [\nabla u + \nabla_y w^\varepsilon - \nabla u^\varepsilon] \,\mathrm{d}x + \sqrt{\varepsilon} C \|\varphi^\varepsilon\|_X \\
&\leq -\alpha \|\nabla u + \nabla_y w^\varepsilon - \nabla u^\varepsilon\|_H^2 + \varepsilon C_\alpha + \alpha \|\varphi^\varepsilon\|_X^2 \\
&\leq C \left\{ \varepsilon + \|u - u^\varepsilon\|_H^2 + \|\varepsilon w^\varepsilon\|_H^2 + \|\varepsilon \nabla_x w^\varepsilon\|_H^2 \right\}.
\end{aligned}$$

Here, $\nabla_x w^\varepsilon$ *denotes the* x*-derivative with respect to* u *– and not* z*. Finally, we arrive at* $\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 \leq C\left\{\varepsilon + \|u^\varepsilon - u\|_H^2\right\}$*. Thus, assuming* $\|u^\varepsilon(0) - u(0)\|_H \leq \varepsilon^{1/2}c$ *for the initial values and applying Gronwall's lemma gives* $\|u^\varepsilon - u\|_{C([0,T];H)} \leq \varepsilon^{1/2}C$*. From that, we also obtain the gradient estimate so that we overall arrive at*

$$\|u^\varepsilon - u\|_{C([0,T];H)} + \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - [\nabla u + \nabla_y U]\|_{\mathrm{L}^2(0,T;\mathbb{H})} \leq \varepsilon^{1/2}C. \tag{2.3.79}$$

Incorporating nonlinear, Lipschitz continuous right-hand sides such as $f(u^\varepsilon)$ *and* $f(u)$ *does not change the error estimate* (2.3.79)*. Indeed, we have*

$$\left|\int_\Omega (f(u^\varepsilon) - f(u)) \cdot \varphi^\varepsilon \,\mathrm{d}x\right| \leq L\|u^\varepsilon - u\|_H \left\{\|u^\varepsilon - u\|_H + \|\varepsilon w^\varepsilon\|_H\right\} \leq C\left\{\varepsilon + \|u^\varepsilon - u\|_H^2\right\}.$$

It should be emphasized that we do not rely on improved time-regularity in this case. We have $\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 = \langle (u^\varepsilon - u)_t, u^\varepsilon - u\rangle_{X^*,X}$ *and do not need the unfolding* $\mathcal{T}_\varepsilon(u_t^\varepsilon)$ *of* u_t^ε *because the periodicity defect occurs only with respect to the corrector function.*

2.4 Summary and Outlook

For the nonlinearly coupled system of reaction-diffusion equations involving different diffusion length scales (2.0.1.P$_\varepsilon$), we have rigorously derived a set of effective equations (2.0.2.P$_0$). Therefore, we assume in Main Theorem II only minimal regularity of the given data. Indeed, the assumptions on the data are such that unique weak solutions of (2.0.1.P$_\varepsilon$) and (2.0.2.P$_0$) exist in the general function space $W(0,T;X)$ and that the data converge in the two-scale sense as $\varepsilon \to 0$. Our results in Section 2.1 and 2.2 significantly generalize similar homogenization results on coupled systems involving slow diffusion such as [Eck05, Muv11, FA*11, Muv13] relying on formal asymptotic expansion or [HJM94, MeM10] using the theory of monotone operators.

Moreover, we prove in Section 2.3 quantitative estimates for the convergence of the solutions. The obtained convergence rate $\varepsilon^{1/4}$ up to the boundary seems to be optimal in comparison to similar results in the literature. We assume only minimal additional regularity of the limiting solutions as in [Gri04, Gri05, FMP12]. However, our choice of the initial values is rather restricted in Main Theorem IIIa. In Main Theorem IIIb, we allow for very general initial values and we can still prove the convergence rate $\varepsilon^{1/6}$. Eventually, one may find a more elaborate approach to derive bounds for $\|u_t^\varepsilon(t)\|_H$ and $\|v_t^\varepsilon(t)\|_H$ so that the latter rate can be improved to $\varepsilon^{1/4}$.

The quantitative estimates for linear elliptic problems in Proposition 2.3.16 and 2.3.17 seem to be novel in the case of classical and slow diffusion. Based on the quantification of the folding mismatch (Proposition 2.3.8), we are able to treat not exactly periodic coefficients such as $\mathbb{D}(x, x/\varepsilon)$ with the periodic unfolding method.

Possible generalizations of the data $\mathbb{D}$ *and* $\mathbb{F}$. It is desirable to relax the growth condition on the reaction term $\mathbb{F}_i$. In the following, we discuss one possible generalization. In the case of classical diffusion, only, the solutions satisfy $u^\varepsilon(t), u(t) \in X$ for a.e. $t \in [0,T]$ and we have the continuous embedding of $X = \mathrm{H}^1(\Omega)$ into $\mathrm{L}^{2^*}(\Omega)$, where 2^* denotes the Sobolev embedding exponent with $2^* \in [1,\infty)$ for $d = 1,2$ and $2^* = 2d/(d-2)$ for $d \geq 3$.

In this case, the presented methods for the existence of solutions and the homogenization limit can be generalized to locally Lipschitz continuous reaction terms with less restricted growth condition such as

$$|\mathbb{F}_1(t,x,y,A)-\mathbb{F}_1(t,x,y,B)| \leq L(1+|A|+|B|)^{\gamma}|A-B| \quad \text{with} \quad \gamma=\tfrac{d}{d+2}-1.$$

For the slow diffusion limit $V(t) \in \mathbb{X} = \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ such a generalization is not immediately possible. Nevertheless, one might be able to prove $V(t) \in \mathrm{L}^{2^*}(\Omega\times\mathcal{Y})\cap\mathbb{X}$ for the limit system (2.0.2.P$_0$). Indeed, for more regular data $\mathbb{D}_2$ and $\mathbb{F}_2$ as in (2.3.2.A1), we can show higher x-regularity $V(t) \in \mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ and we can exploit the embeddings $\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y})) \subset \mathrm{H}^1(\Omega\times\mathcal{Y}) \subset \mathrm{L}^{2^*}(\Omega\times\mathcal{Y})$. However, this seems to be an open problem for the coupled system (2.0.1.P$_\varepsilon$), since v^ε might not be uniformly bounded in $\mathrm{L}^{2^*}(\Omega)$.

Throughout Section 1.1 and Chapter 2, we assume that $\mathbb{D}_i$ and $\mathbb{F}_i$ are continuously differentiable on $(0,T)$ as well as $A \mapsto \mathbb{F}_i(t,x,y,A) \in C^1(\mathbb{R}^m)$, cf. (1.1.5)–(1.1.6). These assumptions are required to deduce the improved time-regularity of solutions in Proposition 1.1.3. However, they are not required for the existence of solutions (Section 1.1) and the derivation of the homogenization limit (Section 2.1). In the course of the regularization in Section 2.2, it seems possible to relax these assumptions to $\mathbb{D}_i, \mathbb{F}_i \in \mathrm{L}^\infty(0,T)$ as well as $A \mapsto \mathbb{F}_i(t,x,y,A) \in C(\mathbb{R}^m)$ by relying on an additional approximation argument.

Our model (2.0.1.P$_\varepsilon$) captures *two different phenomena*, namely the ε-periodic microstructure *and* the different diffusion length scales. The first phenomenon usually occurs when the underlying domain is a porous medium. Since our theory in Section 2.1 and 2.2 allows for spatially discontinuous coefficient functions, we may model periodically distributed heterogeneities in the domain via periodic coefficients. Different diffusion length scales may come into play when some species or some parts of the domain admit different diffusion properties. Natural fields of application for our homogenization theory are reaction and diffusion processes in heterogeneous domains such as concrete carbonation (engineering) or the diffusion of substances through tissue layer (biology).

The phenomenon of slow diffusion is of its own interest and is related to *pattern formation.* The most prominent pattern forming instability is the diffusion-driven Turing instability, first formulated in [Tur52] to describe morphogenesis. The idea of Turing's instability is that a given homogeneous steady state of a reaction-diffusion system is stable when the diffusion terms are neglected and it becomes unstable when the diffusion is reinserted. To this end, the ratio of the diffusion constants should be very large. Turing's instability occurs for instance in the Gray-Scott model, which exhibits a rich spectrum of patterns such as pulses, rings, spots, stripes, traveling waves, self-replicating spots, and spatio-temporal chaos, see e.g. [Kol04, KWW06, KB*09] and references therein. After suitable rescaling, the dimensionless Gray-Scott model reads

$$u_t = \Delta u - u + 1 - uv^2, \quad v_t = \varepsilon^2\Delta v - v + Auv^2, \tag{2.4.1}$$

and it is shown in [Kol04, p. xii] that one of the homogeneous steady states $(\bar{u}_0,\bar{v}_0)$ of (2.4.1) undergoes a Turing instability for $A = 2.3$ and sufficiently small ε as shown in Figure 2.1. It is an open problem whether the effective system (2.0.2.P$_0$) may help studying the Turing instabilities in (2.4.1).

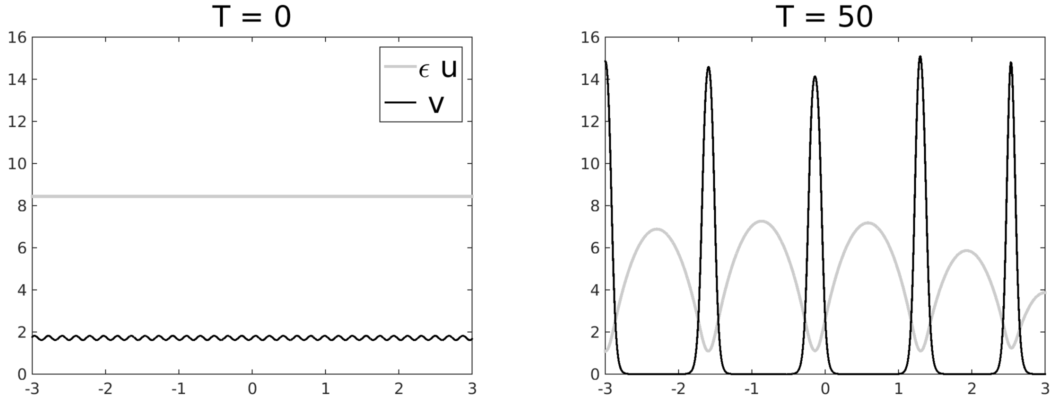

Figure 2.1: Turing pattern in (2.4.1)[1] for the initial value $(u_0, v_0) = (\bar{u}_0, \bar{v}_0 + 0.1 * \sin(x/\varepsilon))$ and $\varepsilon = 0.03$.

Throughout all calculations in Chapter 2, we keep the final time point $T > 0$ fixed in order to apply Gronwall's lemma. It is a further open problem what happens for $T \to \infty$ in the case of slow diffusion (only).

[1]The solution of (2.4.1) is numerically approximated via solving a semi-implicit finite difference scheme implemented in MATLAB.

3 Homogenization of Cahn–Hilliard-type equations via evolutionary Γ-convergence

We consider a family of gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ and address the central question of characterizing the conditions on the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ that guarantee the convergence of solutions u_ε of the multiscale problems associated with $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to solutions of an effective problem in the limit $\varepsilon \to 0$. In particular, as the evolution is entirely driven by functionals we aim for methods based on Γ-convergence and, following [Mie14], call this approach evolutionary Γ-convergence, *E-convergence* for short.

Here, we present two distinct approaches: The first approach is based on the uniform Λ-*convexity* of the driving functionals $\mathcal{E}_\varepsilon$ with respect to the potentials $\mathcal{R}_\varepsilon$, see Subsection 3.1.2 for the definition. In this case we can reformulate the evolution of the system in terms of the *Integrated Evolutionary Variational Estimate (IEVE)*

$$\begin{aligned}&\text{for all } s \text{ and } t \text{ with } 0 \le s < t \text{ and all } w \in \mathrm{dom}(\mathcal{E}_\varepsilon):\\ &\mathrm{e}^{\Lambda(t-s)}\mathcal{R}_\varepsilon(u_\varepsilon(t){-}w) - \mathcal{R}_\varepsilon(u_\varepsilon(s){-}w) \le \mathsf{M}_\Lambda(t{-}s)\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon(t))\big),\end{aligned} \tag{3.0.1}$$

where $\mathsf{M}_\Lambda(r) = \int_0^r \mathrm{e}^{\Lambda\tau}\,\mathrm{d}\tau$. We refer to [AGS05, DaS08, DaS10] for an extensive survey on the topic of Λ-convex gradient systems. Under the general assumptions that the energy functionals Γ-converge to a limit functional with respect to some suitable topology and the dissipation potentials converge continuously to a limit (see (3.1.7)), we can pass to the limit $\varepsilon \to 0$ in the IEVE formulation to derive the effective limit problem.

The second approach to E-convergence is based on the *Energy-Dissipation Principle (EDP)*, which reads

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\,\mathrm{d}t \le \mathcal{E}_\varepsilon(u_\varepsilon(0)). \tag{3.0.2}$$

In contrast to the first approach based on IEVE, the EDP formulation does not rely on any convexity assumptions of the energy functional and follows from the Legendre–Fenchel equivalences and the chain rule. However, we need to additionally impose the *well-preparedness of the initial conditions*, i.e. $u_\varepsilon(0) \to u(0)$ in some sense and $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$, whereas this condition was not needed in IEVE.

Moreover, since the (sub)differential of the driving functional appears in the dual dissipation potential, i.e. $\mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))$, we need an additional condition that guarantees the closedness of the (sub)differential of $\mathcal{E}_\varepsilon$. Combined with the Γ-convergence of the energies and dissipation potentials with respect to suitable topologies in X, the well-preparedness and the closedness of the subdifferential condition allow us to pass to the limit $\varepsilon \to 0$ in (3.0.2) and derive the EDP formulation for the limit system. An important point is that in the later application to homogenization problems the lower liminf estimate for the dissipation potentials with respect to weak convergence in X is not satisfied. Therefore, we have to generalize the abstract E-convergence results via EDP in [Mie14] to fit in our setting.

Having established the two approaches for E-convergence in the abstract case, we apply both methods to rigorously prove a homogenization result for the multiscale Cahn–Hilliard-type equation

$$\partial_t u_\varepsilon = \operatorname{div}\big[M_\varepsilon(x)\nabla\big(\partial_u W_\varepsilon(x,u_\varepsilon) - \operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon)\big)\big]. \tag{3.0.3}$$

The multiple scales are given by the rapidly oscillating coefficient functions $M_\varepsilon(x) = \mathbb{M}(x, x/\varepsilon)$, $A_\varepsilon(x) = \mathbb{A}(x, x/\varepsilon)$, and the potential $W_\varepsilon(x,u) = \mathbb{W}(x, x/\varepsilon, u)$. We show that limits of (subsequences of) solutions to (3.0.3) solve the limiting equation

$$\partial_t u = \operatorname{div}\big[M_{\text{eff}}(x)\nabla\big(\partial_u W_{\text{eff}}(x,u) - \operatorname{div}(A_{\text{eff}}(x)\nabla u)\big)\big], \tag{3.0.4}$$

where the effective coefficient functions $M_{\text{eff}}, A_{\text{eff}}$ are given via the classical unit cell problem and $W_{\text{eff}}(x,u)$ is the usual average of $\mathbb{W}$ over the microscopic cells for fixed u.

The gradient structure $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ of the Cahn–Hilliard-type equation (3.0.3) is well-known, namely X is isomorphic to the dual of H^1-functions with fixed average, $\mathcal{E}_\varepsilon$ is the classical Allen–Cahn energy functional, and $\mathcal{R}_\varepsilon$ is an H^{-1}-norm-like dissipation potential (cf. Subsection 3.2.3). Using the method of two-scale convergence via periodic unfolding, we prove that under suitable assumptions on the potential W_ε the energy functionals $\mathcal{E}_\varepsilon$ Γ-converge to an effective energy functional $\mathcal{E}_0$ with respect to the weak topology on $\mathrm{H}^1(\Omega)$. With the same arguments we can show that the dual dissipation potentials Γ-converge to an effective potential in the *weak topology of* X^* and, thus, by a duality principle for Γ-convergence we obtain the Γ-convergence of the primal dissipation potentials in the *strong topology of* X .

In order to apply the abstract E-convergence results based on IEVE, we assume that the potential W_ε is uniformly λ-convex on $\mathbb{R}$. In that case, we can deduce the uniform Λ-convexity of $\mathcal{E}_\varepsilon$ with Λ related to λ. In particular, in this case the first approach yields the desired homogenized equation (3.0.4).

In the second approach, based on the EDP formulation, we can drop the convexity assumption on W_ε. However, we need to verify closedness properties of the subdifferential of $\mathcal{E}_\varepsilon$. In the concrete case of the Cahn–Hilliard equation in (3.0.3) this follows e.g. from suitable uniform growth estimates for $\partial_u W_\varepsilon$ or uniform λ-convexity of W_ε. We remark that both approaches allow us to consider the classical logarithmic- and double-well potential. However, we show that there are certain examples of potentials that highlight the distinction between the approaches. Last but not least, we highlight that the application of E-convergence via IEVE or EDP is not restricted to equations (such as (3.0.3)). In particular, we can treat systems of equations in the same manner – as long as they admit a gradient structure. The content of this chapter is based on [LiR15].

The structure of Chapter 3 is as follows. In Section 3.1, we introduce abstract gradient systems $(X, \mathcal{E}, \mathcal{R})$ consisting of a separable Hilbert space X, an energy functional $\mathcal{E}$, and a quadratic dissipation potential $\mathcal{R}$. We discuss the notion of evolutionary Γ-convergence in Subsection 3.1.1 and state the two abstract results on the IEVE and EDP formulation in

Subsection 3.1.2 and Subsection 3.1.3, respectively. Section 3.2 is devoted to the homogenization of the Cahn–Hilliard-type equation (3.0.3) and related results in the literature are presented in Subsection 3.2.1. We collect the assumptions on the data in Subsection 3.2.2, explain the gradient structure in Subsection 3.2.3, and derive the Γ-convergence of the energy and dissipation functionals in Subsection 3.2.4. Here, we restrict ourselves for simplicity to classes of potentials satisfying a suitable growth condition. Finally, we apply the abstract results of Subsection 3.1.1 based on IEVE and EDP to the concrete setting in Subsection 3.2.5 and 3.2.6, respectively. In Subsection 3.2.7, we present exemplary potentials W_ε, that fit into our theory. Finally, we conclude the paper in Section 3.3 by discussing the benefits and differences of the two approaches via IEVE and EDP, respectively. Moreover, we compare our E-convergence results with that of [SaS04].

3.1 Abstract gradient systems

A gradient system is a triple $(X, \mathcal{E}, \mathcal{R})$ consisting of a separable Hilbert space X, a proper and lower semicontinuous driving functional $\mathcal{E} : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, and a quadratic dissipation potential $\mathcal{R} : X \to [0, \infty)$. The latter means that $\mathcal{R}$ is of the form $\mathcal{R}(v) = \frac{1}{2}\langle Gv, v\rangle$ with $\langle\cdot,\cdot\rangle$ denoting the dual pairing between X and its dual X^* (which we do not identify to distinguish between velocities and forces) and $G \in \mathrm{Lin}(X, X^*)$ is symmetric and positive definite. In particular, we assume that $\mathcal{R}$ satisfies

$$\exists\, \alpha, \beta > 0 : \quad \tfrac{\alpha}{2}\|v\|_X^2 \le \mathcal{R}(v) \le \tfrac{\beta}{2}\|v\|_X^2 \quad \text{for all } v \in X. \tag{3.1.1}$$

The gradient-flow equation associated with $\mathcal{E}$ and $\mathcal{R}$ is now given in terms of the force balance, also called Biot's equation, which reads

$$0 \in \mathrm{D}\mathcal{R}(\dot{u}(t)) + \partial_X \mathcal{E}(u(t)), \quad u(0) = u_0, \tag{3.1.2}$$

where $\partial_X \mathcal{E}(u) \subset X^*$ denotes a suitable notion of a set-valued subdifferential of $\mathcal{E}$. Let us remark that the right notion of subdifferential, e.g. convex, Fréchet, or strong/weak limiting subdifferential, is dictated by the concrete problem. On the one hand, it has to be "big" enough such that all relevant limits are contained. On the other hand, it has to be "small" enough to satisfy a chain rule condition (see below). We refer to [RoS06] for a discussion of sufficient conditions on $\mathcal{E}$, its subdifferential $\partial_X\mathcal{E}$, and the data u_0 that guarantee the existence of solutions of (3.1.2), see also Remark 3.1.1. In the following, we always assume that solutions $u \in \mathrm{H}^1(0,T;X)$ of the force-balance formulation in (3.1.2) exist.

With the primal dissipation potential $\mathcal{R}$ we can associate the dual dissipation potential $\mathcal{R}^* : X^* \to [0, \infty)$, which is given via the Legendre transform, i.e.

$$\mathcal{R}^*(\xi) := \sup\big\{\langle \xi, v\rangle - \mathcal{R}(v) \,|\, v \in X\big\}.$$

In particular, we have that $\mathcal{R}^*(\xi) := \frac{1}{2}\langle \xi, G^{-1}\xi\rangle$ and the estimates $\frac{\alpha^*}{2}\|\xi\|_{X^*}^2 \le \mathcal{R}^*(\xi) \le \frac{\beta^*}{2}\|\xi\|_{X^*}^2$ are satisfied for all $\xi \in X^*$, where $\alpha^* = 1/\beta$ and $\beta^* = 1/\alpha$.

For the driving functional $\mathcal{E}$, we assume that there exists a reflexive Banach space $Z \subset X$ such that the embedding is compact and

$$\exists\, c, C > 0,\ q \ge 1 :\ \mathcal{E}(u) \ge c\|u\|_Z^q - C \ \text{ for all } u \in Z. \tag{3.1.3}$$

As usual, we extend $\mathcal{E}$ to the bigger space X by setting $\mathcal{E}(u) = +\infty$ for $u \in X \setminus Z$.

Finally, we make the crucial assumption that $\partial_X \mathcal{E}$ satisfies a chain rule condition: If $u \in \mathrm{H}^1(0,T;X)$, $\xi \in \mathrm{L}^2(0,T;X^*)$ is such that $\xi(t) \in \partial_X\mathcal{E}(u(t))$ for a.a. $t \in [0,T]$, and $t \mapsto \mathcal{E}(u(t))$ is bounded, then it is also absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) = \langle \xi(t), \dot{u}(t)\rangle \quad \text{for a.e. } t \in [0,T]. \tag{3.1.4}$$

Remark 3.1.1. *Our setting can be cast in the framework of [RoS06] by considering the Hilbert space X with norm $\|v\|_G^2 = \langle Gv, v\rangle$ and the corresponding subdifferential $\partial_G\mathcal{E} = G^{-1}\partial_X\mathcal{E} \subset X$, meaning that $v \in \partial_G\mathcal{E}(u)$ if and only if $Gv \in \partial_X\mathcal{E}(u)$.*

If $u_0 \in \mathrm{dom}(\mathcal{E})$, the coercivity and the chain rule conditions in (3.1.3) *and* (3.1.4) *are satisfied, then solutions $u \in \mathrm{H}^1(0,T;X)$ of* (3.1.2) *exist according to [RoS06, Thm. 3] with $\partial_X\mathcal{E}$ being the strong-weak limiting subdifferential. Indeed, assuming additional continuity properties of $\mathcal{E}$ (continuity along sequences of equi-bounded slope) the chain rule condition* (3.1.4) *can be weakened such that $t \mapsto \mathcal{E}(u(t))$ is a.e. equal to a function of bounded variation $\varphi : [0,T] \to \mathbb{R}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \langle \xi(t), \dot{u}(t)\rangle$.*

3.1.1 Evolutionary Γ-convergence for abstract gradient systems

For a parameter $\varepsilon \in [0,1]$, we consider a family of gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where X, $\mathcal{E}_\varepsilon$, and $\mathcal{R}_\varepsilon$ are as above for each ε. Following [Mie14, Def. 2.10], we define the notion of evolutionary Γ-convergence with or without well-prepared initial conditions – *E-convergence* respective *well-prepared E-convergence* for short.

Definition 3.1.2 (E-convergence)**.** *For $\varepsilon > 0$, let $u_\varepsilon : [0,T] \to X$ be a solution of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ in the sense of* (3.1.2) *and assume that $u_\varepsilon(0) \to u_0$ in X. We say that $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$* E-converges *to $(X, \mathcal{E}_0, \mathcal{R}_0)$ if there exists a solution $u : [0,T] \to X$ of $(X, \mathcal{E}_0, \mathcal{R}_0)$ with $u(0) = u_0$ and a subsequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k}(t) \to u(t)$ in X as well as $\mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}(t)) \to \mathcal{E}_0(u(t))$ for all $t \in (0,T]$.*

If we need to impose additionally $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u_0) < \infty$, we say that $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges with well-prepared initial conditions *to $(X, \mathcal{E}_0, \mathcal{R}_0)$.*

In the upcoming subsections, we prove two abstract E-convergence results: In Theorem 3.1.5 we impose a uniform Λ-convexity condition on $\mathcal{E}_\varepsilon$ to show the E-convergence of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ using an equivalent formulation based on evolutionary variational inequalities and without well-preparedness of the initial conditions. Secondly, we prove the same result in Theorem 3.1.6 assuming well-preparedness and a closedness property of the subdifferentials instead of the Λ-convexity condition by passing to the limit in the energy-dissipation formulation of (3.1.2). Both approaches are based on the Γ-convergence of the functionals whose definition we recall here.

Definition 3.1.3 (Γ- and Mosco convergence)**.** *On a reflexive Banach space X we say that the functionals $\mathcal{E}_\varepsilon$* Γ-converge *to $\mathcal{E}_0$ in the* weak (resp. strong) *topology on X, and* *write* $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0$ *(resp.* $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$*), if the following two estimates are satisfied*

(i) liminf estimate

$\forall\, u_\varepsilon \rightharpoonup u$ *(resp.* $u_\varepsilon \to u$*)* : $\quad \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u)$;

(ii) limsup estimate (existence of recovery sequences)

$\forall\, \widehat{u}\ \exists\, \widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ *(resp.* $\widehat{u}_\varepsilon \to \widehat{u}$*)* : $\quad \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \leq \mathcal{E}_0(\widehat{u})$.

We say that $\mathcal{E}_\varepsilon$ converges in the sense of Mosco *to* $\mathcal{E}_0$*, written* $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$*, if (i) holds with respect to the weak convergence in* X *and (ii) is satisfied with respect to the strong convergence, i.e. strongly converging recovery sequences exist.*

Let the systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ satisfy the assumptions (3.1.1) and (3.1.3) uniformly with respect to ε, i.e. there exist constants $\alpha, \beta, C, c > 0$, a reflexive Banach space $Z \subset X$ compactly, and $q \geq 1$, all independent of ε, such that

$$\forall\, \varepsilon \in [0,1]: \quad \begin{cases} \forall\, v \in X: & \frac{\alpha}{2}\|v\|_X^2 \leq \mathcal{R}_\varepsilon(v) \leq \frac{\beta}{2}\|v\|_X^2; \\ \forall\, u \in X: & \mathcal{E}_\varepsilon(u) \geq c\|u\|_Z^q - C. \end{cases} \tag{3.1.5}$$

Moreover, we assume in the following that the driving functionals $\mathcal{E}_\varepsilon$ and the dissipation potentials $\mathcal{R}_\varepsilon$ Γ-converge in the strong sense on X, respectively, namely

$$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0 \text{ in } X \quad \text{and} \quad \mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 \text{ in } X. \tag{3.1.6}$$

Finally, in the uniform Λ-convex case in Subsection 3.1.2 we will additionally assume that the dissipation potentials $\mathcal{R}_\varepsilon$ *converge continuously* along strongly converging sequences in X, denoted $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$, i.e.

$$\forall\, u_\varepsilon \to u \text{ in } X: \quad \lim_{\varepsilon\to 0} \mathcal{R}_\varepsilon(u_\varepsilon) = \mathcal{R}_0(u). \tag{3.1.7}$$

Since Z is compactly embedded in X and the family $\mathcal{E}_\varepsilon$ is equi-coercive on Z, the weak Γ-convergence on Z is equivalent to Mosco convergence on X. Moreover, the strong Γ-convergence on X of the dissipation potentials $\mathcal{R}_\varepsilon$ is equivalent to the weak Γ-convergence of $\mathcal{R}_\varepsilon^*$ on X^* due to the continuity properties of the Legendre transform. We collect these two results in the following proposition.

Proposition 3.1.4. *(a) [Mie14, Prop. 2.5] Assuming the equi-coercivity in* (3.1.5) *and the compact embedding of* Z *in* X *the following is equivalent*

$$\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0 \text{ in } Z \quad \Longleftrightarrow \quad \mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_\varepsilon \text{ in } X. \tag{3.1.8}$$

(b) [Att84, pp. 271] For $\varepsilon \in [0,1]$ *let* $\mathcal{R}_\varepsilon^*$ *denote the Legendre transform of* $\mathcal{R}_\varepsilon$*, then*

$$\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 \text{ in } X \quad \Longleftrightarrow \quad \mathcal{R}_\varepsilon^* \xrightharpoonup{\Gamma} \mathcal{R}_0^* \text{ in } X^*. \tag{3.1.9}$$

3.1.2 A convergence result based on variational inequalities

In this subsection, we prove the first abstract Γ-convergence result for the gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ in the case that $\mathcal{E}_\varepsilon$ is uniformly Λ-convex with respect to the dissipation potential $\mathcal{R}_\varepsilon$, i.e. we assume that there exists a constant $\Lambda \in \mathbb{R}$, independent of ε, such that

$$u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda \mathcal{R}_\varepsilon(u) \quad \text{is convex.} \tag{3.1.10}$$

If the driving functional $\mathcal{E}_\varepsilon$ is Λ-convex with respect to $\mathcal{R}_\varepsilon$ in the sense of (3.1.10) we obtain the equivalent formulation of the (differential) gradient-flow equation in (3.1.2) as an evolutionary variational estimate EVE. We recall that the *Fréchet subdifferential* $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon : X \rightrightarrows X^*$ is defined via

$$\partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u) := \Big\{ \xi \in X^* \,\Big|\, \liminf_{w \to u} \frac{\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u) - \langle \xi, w{-}u\rangle}{\|w{-}u\|_X} \geq 0 \Big\} \tag{3.1.11}$$

and is in general multi-valued. In particular, in the Λ-convex case we have that $\xi \in \partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u)$ for $u \in X$ if and only if

$$\text{for all } w \in X: \quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u) + \langle \xi, w{-}u\rangle + \Lambda \mathcal{R}_\varepsilon(w{-}u). \tag{3.1.12}$$

Moreover, if $\mathcal{E}_\varepsilon$ is Λ-convex $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon$ satisfies the chain rule condition (see e.g. [Bré73, Lem. 3.3]) as well as the strong-weak closedness condition, cf. Proposition 3.1.7.

Using this convexity estimate and the gradient-flow equation in (3.1.2) for $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ we arrive at the Evolutionary Variational Estimate (EVE)

$$\forall\, t > 0,\, w \in X: \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_\varepsilon(u(t){-}w) + \Lambda \mathcal{R}_\varepsilon(u(t){-}w) \leq \mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t)), \tag{3.1.13}$$

which corresponds to the Hilbert space version of Bénilan's weak formulation [Bén72] in the case $\Lambda = 0$, see also [AGS05, Ch. 4] and [DaS10]. Multiplying the estimate in (3.1.13) with $\mathrm{e}^{\Lambda t}$ and integrating over an interval $[r, s]$, for $s > r \geq 0$, gives the equivalent Integrated Evolutionary Variational Estimate (IEVE)

$$\forall\, w \in X: \quad \mathrm{e}^{\Lambda(s-r)}\mathcal{R}_\varepsilon(u_\varepsilon(s){-}w) - \mathcal{R}_\varepsilon(u_\varepsilon(r){-}w) \leq \mathsf{M}_\Lambda(s{-}r)\Big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon(s))\Big) \tag{3.1.14}$$

with $\mathsf{M}_\Lambda(\tau) = (\mathrm{e}^{\Lambda\tau}{-}1)/\Lambda$ for $\Lambda \neq 0$ and $\mathsf{M}_0(\tau) = \tau$, see also [DaS08, Prop. 3.1]. Note, that this formulation is only written in terms of functionals and no derivatives appear.

We state the main result of this subsection on the evolutionary Γ-convergence of the gradient system $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ that can be found in [Mie15]. Theorem 3.1.5 only differs from [Mie15, Thm. 3.2] with respect to the assumption $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ instead of $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$, however the proof still applies. Note that this is a variant of [DaS10, Thm. 2.17], see also [Mie14].

Theorem 3.1.5. *Let $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ satisfy the equi-coercivity conditions in* (3.1.5) *and assume that $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ and $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$ in X. Assume moreover that the convexity property in* (3.1.10) *is satisfied and that the initial conditions are such that $u_\varepsilon(0) \to u(0)$ in X with $u(0) \in \overline{\mathrm{dom}(\mathcal{E}_0)}^X$. Then, $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges to $(X, \mathcal{E}_0, \mathcal{R}_0)$ and the limit $t \mapsto u(t)$ satisfies*

$$\forall\, t > 0,\, w \in X: \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_0(u(t){-}w) + \Lambda \mathcal{R}_0(u(t){-}w) \leq \mathcal{E}_0(w) - \mathcal{E}_0(u(t)). \tag{3.1.15}$$

Moreover, for each $t \in (0, T]$ the energies converge, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$.

Proof. *Step 1: A priori estimates.* Since $\mathcal{E}_0$ is a proper functional we can find a recovery sequence $\widehat{w}_\varepsilon \in X$ with $\mathcal{E}_\varepsilon(\widehat{w}_\varepsilon) \le C < \infty$. Hence, for $r = 0$ we get from from (3.1.14)

$$\mathrm{e}^{\Lambda s}\mathcal{R}_\varepsilon(u_\varepsilon(s){-}\widehat{w}_\varepsilon) + \mathsf{M}_\Lambda(s)\mathcal{E}_\varepsilon(u_\varepsilon(s)) \le \mathcal{R}_\varepsilon(u_\varepsilon(0){-}\widehat{w}_\varepsilon) + \mathsf{M}_\Lambda(s)\mathcal{E}_\varepsilon(\widehat{w}_\varepsilon) \le C < \infty. \quad (3.1.16)$$

Due to the positivity of $\mathcal{R}_\varepsilon$ and the estimate $0 < m_0 \le \mathsf{M}_\Lambda(s)$ for all $0 < t_0 \le s \le T$, we obtain

$$\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(u_\varepsilon(t)) < \infty \quad \text{for all } t \in [t_0, T]. \quad (3.1.17)$$

Hence, by the equi-coercivity of $\mathcal{E}_\varepsilon$ we obtain a uniform bound for u_ε in $\mathrm{L}^\infty([t_0,T];Z)$.

Let us consider a partition $t_k = t_0 + k\tau_N$ with $k = 0, \ldots, N$ and $\tau_N = (T{-}t_0)/N$ for $N \in \mathbb{N}$. Replacing s and r with t_k and t_{k-1}, respectively, as well as taking $w = u_\varepsilon(t_{k-1})$ in (3.1.14), we arrive at

$$\mathcal{R}_\varepsilon(u_\varepsilon(t_k){-}u_\varepsilon(t_{k-1})) \le \mathrm{e}^{-\Lambda\tau_N}\mathsf{M}_\Lambda(\tau_N)\Big(\mathcal{E}_\varepsilon(u_\varepsilon(t_{k-1})) - \mathcal{E}_\varepsilon(u_\varepsilon(t_k))\Big).$$

Summing over $k = 1, \ldots, N$ and taking the limit $N \to \infty$ gives the standard estimate

$$\int_{t_0}^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon(s))\,\mathrm{d}s \le \mathcal{E}_\varepsilon(u_\varepsilon(t_0)) - \mathcal{E}_\varepsilon(u_\varepsilon(T)).$$

By (3.1.17) and the equi-coercivity of $\mathcal{R}_\varepsilon$, we obtain a uniform bound in $\mathrm{C}^{1/2}([t_0,T];X)$ for all $t_0 \in (0,T)$. By Arzelà–Ascoli's theorem, we find a (not relabeled) subsequence such that $u_\varepsilon(t) \rightharpoonup u_*(t)$ in Z for all $t > 0$ and by the compact embedding $Z \subset X$ also strongly in X. For $t = 0$, we set $u_*(0) = u(0)$.

Step 2: Limit passage in IEVE. To pass to the limit in (3.1.14) we take an arbitrary test state $\widehat{w}$ and choose a recovery sequence $\widehat{w}_\varepsilon$ such that $\widehat{w}_\varepsilon \to \widehat{w}$ in X and $\mathcal{E}_\varepsilon(\widehat{w}_\varepsilon) \to \mathcal{E}_0(\widehat{w})$. Using the lim inf-estimate for $\mathcal{E}_\varepsilon$ and the continuous convergence of $\mathcal{R}_\varepsilon$ in X yields for all $0 \le r < s$

$$\mathrm{e}^{\Lambda(s-r)}\mathcal{R}_0(u_*(s){-}\widehat{w}) - \mathcal{R}_0(u_*(r){-}\widehat{w}) \le \mathsf{M}_\Lambda(s{-}r)(\mathcal{E}_0(\widehat{w}){-}\mathcal{E}_0(u_*(s))). \quad (3.1.18)$$

Thus, u_* is a solution of the variational inequality (3.1.14) for $\varepsilon = 0$. However, it remains to show that $\lim_{s\to 0+} u_*(s) = u(0)$. For this, let $r = 0$ and $\widehat{w} \in \mathrm{dom}(\mathcal{E}_0)$, and consider the limit $s \to 0^+$ in (3.1.14) for $\varepsilon = 0$

$$\lim_{s\to 0^+} \mathrm{e}^{\Lambda s}\mathcal{R}_0(u_*(s){-}\widehat{w}) - \mathcal{R}_0(u(0){-}\widehat{w}) \le \lim_{s\to 0^+} \mathsf{M}_\Lambda(s)(\mathcal{E}_0(\widehat{w}) - \inf \mathcal{E}_0) = 0,$$

since $\mathsf{M}_\Lambda(s) = O(s)$. Thus, we have $\lim_{s\to 0^+} \|u_*(s){-}\widehat{w}\|_X \le \|u(0){-}\widehat{w}\|_X$ for all $\widehat{w} \in \mathrm{dom}(\mathcal{E}_0)$. Taking an approximating sequence $\widehat{w}_k \to u(0) \in \overline{\mathrm{dom}(\mathcal{E}_0)}^X$ with $\widehat{w}_k \in \mathrm{dom}(\mathcal{E}_0)$ we conclude $u_*(s) \to u(0)$ as $s \to 0^+$.

Step 3: Convergence of the energies. It remains to show that $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u_*(t))$ for all $t \in (0,T]$. For this let $\|\cdot\|^2_{\mathcal{R}_\varepsilon} = 2\mathcal{R}_\varepsilon(\cdot)$ and define the slope $e_\varepsilon(u) := \inf\{\|\xi\|_{\mathcal{R}^*_\varepsilon} \,|\, \xi \in \partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u)\}$ for $\varepsilon \in [0,1]$. Due to the Λ-convexity of $\mathcal{E}_\varepsilon$ we have for all $t > 0$ the lower bound

$$\mathcal{E}_\varepsilon(w) \ge \mathcal{E}_\varepsilon(u_\varepsilon(t)) - e_\varepsilon(u_\varepsilon(t))\big\|w{-}u_\varepsilon(t)\big\|_{\mathcal{R}_\varepsilon} + \Lambda\mathcal{R}_\varepsilon(w{-}u_\varepsilon(t)). \quad (3.1.19)$$

The lower bound in (3.1.16) can be improved in the following way (see [DaS10, Eq. (2.9)])

$$\mathrm{e}^{\Lambda t}\mathcal{R}_\varepsilon(u_\varepsilon(t){-}w_\varepsilon) + \mathsf{M}_\Lambda(t)\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \frac{\mathsf{M}_\Lambda(t)^2}{2}e_\varepsilon(u_\varepsilon(t))^2 \le C.$$

Hence, as above we can find a constant $C(t_0)$ such that the slopes are uniformly bounded for all $t \in [t_0, T]$ with $t_0 > 0$ and all $\varepsilon \in [0,1]$. Fixing $t \in [t_0, T]$ and choosing a recovery sequence $\widehat{u}_\varepsilon \to u_*(t)$ in X gives with (3.1.19)

$$\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \geq \mathcal{E}_\varepsilon(u_\varepsilon(t)) - C(t_0)\big\|\widehat{u}_\varepsilon - u_\varepsilon(t)\big\|_{\mathcal{R}_\varepsilon} + \Lambda \mathcal{R}_\varepsilon(\widehat{u}_\varepsilon - u_\varepsilon(t)).$$

Hence, using $u_\varepsilon(t) \to u_*(t)$ we can pass to the limit $\varepsilon \to 0$ and obtain the estimate $\mathcal{E}_0(u_*(t)) \geq \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t))$. The opposite estimate follows from the Γ-convergence of $\mathcal{E}_\varepsilon$ and we conclude that $\mathcal{E}_0(u_*(t)) = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t))$ for all $t \in (0, T]$. □

3.1.3 A convergence result for the energy-dissipation principle

We establish the second approach for E-convergence based on the energy-dissipation principle in (3.0.2). Indeed, the latter gives an equivalent formulation of (3.1.2) if the chain rule (3.1.4) is satisfied. The crucial point is that for general convex potentials $\Psi : X \to [0, \infty]$ the Legendre–Fenchel equivalences hold, namely

$$\forall\, v \in X,\, \xi \in X^* : \quad \xi \in \partial\Psi(v) \;\Leftrightarrow\; v \in \partial\Psi^*(\xi) \;\Leftrightarrow\; \Psi(v) + \Psi^*(\xi) \leq \langle \xi, v\rangle.$$

Hence, assuming that $u_\varepsilon \in \mathrm{H}^1(0,T;X)$ is a solution of the differential formulation (3.1.2) with respect to $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, we have $\mathcal{R}_\varepsilon(\dot{u}_\varepsilon) + \mathcal{R}^*_\varepsilon(\xi_\varepsilon) \leq \langle \xi_\varepsilon, \dot{u}_\varepsilon\rangle$ a.e. in $[0,T]$, where $\xi_\varepsilon \in \mathrm{L}^2(0,T;X^*)$ satisfies $\xi_\varepsilon(t) \in \partial_X \mathcal{E}_\varepsilon(u_\varepsilon(t))$ for a.a. $t \in [0,T]$. Using the chain rule (3.1.4), we obtain the energy-dissipation principle (EDP) after integrating over $[0,T]$

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon(s)) + \mathcal{R}^*_\varepsilon(\xi_\varepsilon(s))\,\mathrm{d}s \leq \mathcal{E}_\varepsilon(u_\varepsilon(0)), \quad \xi_\varepsilon(t) \in \partial_X \mathcal{E}_\varepsilon(u_\varepsilon(t)). \tag{3.1.20}$$

Conversely, if (3.1.20) is satisfied we easily check that u_ε also solves the differential formulation (3.1.2) (see e.g. [Mie14, Thm. 3.2]). Moreover, note that estimate (3.1.20) is in fact an equality. Indeed, by the elementary estimate $\mathcal{R}_\varepsilon(v) + \mathcal{R}^*_\varepsilon(\xi) \geq \langle \xi, v\rangle$ and the chain rule (3.1.4), we obtain

$$\begin{aligned} &\text{if } \widehat{u} \in \mathrm{H}^1(0,T;X),\ \widehat{\xi} \in \mathrm{L}^2(0,T;X^*),\ \widehat{\xi}(t) \in \partial_X \mathcal{E}_\varepsilon(\widehat{u}(t)) \text{ for a.a. } t \in [0,T],\\ &\text{then } \mathcal{E}_\varepsilon(\widehat{u}(t)) + \textstyle\int_s^t \mathcal{R}_\varepsilon(\dot{\widehat{u}}) + \mathcal{R}^*_\varepsilon(\widehat{\xi}\,)\,\mathrm{d}r \geq \mathcal{E}_\varepsilon(\widehat{u}(s)) \text{ for all } 0 \leq s < t \leq T. \end{aligned} \tag{3.1.21}$$

The following result, being a slight variation of [Mie14, Thm. 3.3 & 3.6], based on (3.1.20) is in the spirit of Sandier & Serfaty's approach [SaS04, Ser11] (see Section 3.3 for a comparison). Note that in contrast to the subsequent section, we do not require any convexity properties of $\mathcal{E}_\varepsilon$ and the continuous convergence of $\mathcal{R}_\varepsilon$ to $\mathcal{R}_0$ can be relaxed to strong Γ-convergence. However, we have to impose additionally well-preparedness of the initial conditions and a closedness condition on the subdifferential of $\mathcal{E}_\varepsilon$ to be able to identify the limit formulation. The latter is formulated such that it fits into our general setting and can be weakened in more concrete situations, see e.g. Proposition 3.1.7. The novelity of the following proof is to use time-discretizations for the solutions and Jensen's inequality in order to derive $\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon)\,\mathrm{d}t \geq \int_0^T \mathcal{R}_0(\dot{u})\,\mathrm{d}t$ although $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ *strongly* and $\dot{u}_\varepsilon \rightharpoonup \dot{u}$ *weakly* in X, only.

Theorem 3.1.6. *Let $Z \subset X$ compactly as well as $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ satisfy the assumptions* (3.1.5) *and* (3.1.6) *on equi-coercivity and Γ-convergence. Moreover, we assume that the initial conditions are well-prepared, i.e.*

$$u_\varepsilon(0) \to u(0) \text{ in } X \quad \text{and} \quad \mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0)) < \infty, \tag{3.1.22}$$

and that the subdifferential $\partial_X \mathcal{E}_\varepsilon$ is closed in the sense

$$\left.\begin{aligned} \widehat{u}_\varepsilon \overset{*}{\rightharpoonup} \widehat{u} \text{ in } \mathrm{L}^\infty(0,T;Z),\ \widehat{u}_\varepsilon \rightharpoonup \widehat{u} \text{ in } \mathrm{H}^1(0,T;X), \\ \widehat{\xi}_\varepsilon \rightharpoonup \widehat{\xi} \text{ in } \mathrm{L}^2(0,T;X^*), \\ \widehat{\xi}_\varepsilon(t) \in \partial_X \mathcal{E}_\varepsilon(\widehat{u}_\varepsilon(t)) \text{ f.a.a. } t \in [0,T] \end{aligned}\right\} \Rightarrow \begin{aligned} &\text{f.a.a. } t \in [0,T]: \\ &\widehat{\xi}(t) \in \partial_X \mathcal{E}_0(\widehat{u}(t)). \end{aligned} \tag{3.1.23}$$

Then, we have the well-prepared E-convergence of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to $(X, \mathcal{E}_0, \mathcal{R}_0)$. In particular, the limit $t \mapsto u(t)$ satisfies

$$\mathcal{E}_0(u(T)) + \int_0^T \mathcal{R}_0(\dot{u}(t)) + \mathcal{R}_0^*(\xi(t))\,\mathrm{d}t \le \mathcal{E}_0(u(0)), \quad \xi(t) \in \partial_X \mathcal{E}_0(u(t)). \tag{3.1.24}$$

Moreover, for each $t \in [0,T]$ the energies converge, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$.

Proof. *Step 1: Uniform bounds.* Using the well-preparedness of the initial conditions (3.1.22), we find a constant $C > 0$ such that $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \le C$. Since the energy-dissipation estimate (3.1.20) is satisfied we immediately get $\int_0^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \le C$ such that by the uniform coercivity of $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\varepsilon^*$ we obtain uniform bounds for $\|\dot{u}_\varepsilon\|_{\mathrm{L}^2(0,T;X)}$ and $\|\xi_\varepsilon\|_{\mathrm{L}^2(0,T;X^*)}$.

Moreover, the upper bound (3.1.21) holds for the time-reversed curve $\widehat{u}_\varepsilon(t) = u_\varepsilon(T-t)$. Due to the invariance of the dissipation potentials with respect to this transformation we obtain for $t = T$

$$\mathcal{E}_\varepsilon(u_\varepsilon(s)) + \int_s^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}r \ge \mathcal{E}_\varepsilon(u_\varepsilon(T-s)).$$

Thus, the coercivity (3.1.5), the well-preparedness (3.1.22), and the uniform bound for the total dissipation imply $\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_Z \le C$. In particular, we have shown the uniform a priori bounds

$$\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;Z)} + \|u_\varepsilon\|_{\mathrm{H}^1(0,T;X)} + \|\xi_\varepsilon\|_{\mathrm{L}^2(0,T;X^*)} \le C. \tag{3.1.25}$$

Step 2: Convergent subsequence. Due to (3.1.25) we can extract a converging subsequence (not relabeled) giving

$$u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } \mathrm{L}^\infty(0,T;Z), \quad u_\varepsilon \rightharpoonup u \text{ in } \mathrm{H}^1(0,T;X), \quad \text{and } \xi_\varepsilon \rightharpoonup \xi \text{ in } \mathrm{L}^2(0,T;X^*). \tag{3.1.26}$$

Moreover, by Arzelà–Ascoli's theorem and the compact embedding $Z \subset X$, we have

$$\forall\, t \in [0,T]: \quad u_\varepsilon(t) \rightharpoonup u(t) \text{ in } Z \text{ and } u_\varepsilon(t) \to u(t) \text{ in } X. \tag{3.1.27}$$

Step 3: Passing to the limit. We show that the limit u satisfies (3.1.24). Note that the right-hand side in (3.1.20) converges because of the well-preparedness of the initial data. Moreover, from $u_\varepsilon(T) \to u(T)$ in X and $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in X (cf. (3.1.6) and (3.1.8)), we obtain

$\mathcal{E}_0(u(T)) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(T))$. Thus, it remains to prove a lower estimate for the total dissipation, namely

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \ge \int_0^T \mathcal{R}_0(\dot u) + \mathcal{R}_0^*(\xi)\,\mathrm{d}t. \tag{3.1.28}$$

For this, let $0 = t_0^N < t_1^N < \ldots < t_N^N = T$ denote an equidistant partition of the interval $[0,T]$ with time step $\tau_N = T/N$, $N \in \mathbb{N}$. Then, Jensen's inequality yields

$$\begin{aligned}\int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t &= \sum_{k=1}^N \int_{t_{k-1}^N}^{t_k^N} \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \\ &\ge \sum_{k=1}^N \tau_N \Big\{\mathcal{R}_\varepsilon\Big(\tfrac{1}{\tau_N}\textstyle\int_{t_{k-1}}^{t_k} \dot u_\varepsilon\,\mathrm{d}t\Big) + \mathcal{R}_\varepsilon^*\Big(\tfrac{1}{\tau_N}\textstyle\int_{t_{k-1}}^{t_k} \xi_\varepsilon\,\mathrm{d}t\Big)\Big\}.\end{aligned} \tag{3.1.29}$$

We introduce $V_k^{N,\varepsilon} := (u_\varepsilon(t_k^N) - u_\varepsilon(t_{k-1}^N))/\tau_N \in X$ and $\Xi_k^{N,\varepsilon} := \frac{1}{\tau_N}\int_{t_{k-1}^N}^{t_k^N} \xi_\varepsilon\,\mathrm{d}s \in X^*$ for $k = 1,\ldots,N$. Using $u_\varepsilon(t_k^N) \to u(t_k^N)$ in X and $\xi_\varepsilon \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;X^*)$, we obtain

$$V_k^{N,\varepsilon} \to V_k^N := \frac{u(t_k^N) - u(t_{k-1}^N)}{\tau_N} \text{ in } X \quad \text{and} \quad \Xi_k^{N,\varepsilon} \rightharpoonup \Xi_k^N := \frac{1}{\tau_N}\int_{t_{k-1}^N}^{t_k^N} \xi\,\mathrm{d}s \text{ in } X^*.$$

Hence, $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ in X and $\mathcal{R}_\varepsilon^* \xrightharpoonup{\Gamma} \mathcal{R}_0^*$ in X^* (cf. (3.1.6) and (3.1.9)) yield the lower estimate

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \ge \sum_{k=1}^N \tau_N \Big\{\mathcal{R}_0(V_k^N) + \mathcal{R}_0^*(\Xi_k^N)\Big\}. \tag{3.1.30}$$

Next, we aim to pass to the limit $N \to \infty$. Let $u_N \in \mathrm{H}^1(0,T;X)$ denote the piecewise affine interpolant such that $u_N(t_k^N) = u(t_k^N)$ and $\dot u_N(t) = V_k^N$ for $t \in (t_{k-1}^N, t_k^N]$. Moreover, we denote by $\xi_N \in \mathrm{L}^2(0,T;X^*)$ the piecewise constant interpolant satisfying $\xi_N(t) = \Xi_k^N$ for $t \in (t_{k-1}^N, t_k^N]$. We easily check that $u_N \rightharpoonup u$ in $\mathrm{H}^1(0,T;X)$ and $\xi_N \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;X^*)$ such that by Ioffe's lower semicontinuity result [Iof77], we are able to pass to the limit $N \to \infty$ in (3.1.30) and finally arrive at

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \ge \int_0^T \mathcal{R}_0(\dot u) + \mathcal{R}_0^*(\xi)\,\mathrm{d}t.$$

By the closedness of the subdifferentials (3.1.23), we immediately have $\xi(t) \in \partial_X \mathcal{E}_0(u(t))$ for a.a. $t \in [0,T]$. Thus, we have shown that u solves the limiting energy-dissipation formulation (3.1.24).

Step 4: Convergence of the energies. Recalling the derivation of (3.1.20) resp. (3.1.24) via the chain rule, we indeed have equality in (3.1.24) on each time interval. Since we have the convergence of the initial energies $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$ by (3.1.22), the lim inf-estimate derived in Step 3 must actually attain a limit. Hence, we have for all $t \in [0,T]$

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t)) \text{ and } \int_0^t \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \to \int_0^t \mathcal{R}_0(\dot u) + \mathcal{R}_0^*(\xi)\,\mathrm{d}t.$$

Thus, we have established the well-prepared E-convergence of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. $\square$

Note, that the usual strong-weak closedness of the graph of the subdifferential $\partial_X\mathcal{E}_\varepsilon$ in the sense of

$$\left.\begin{aligned} u_\varepsilon \to u \text{ in } X,\ \mathcal{E}_\varepsilon(u_\varepsilon) \to e_0,\\ \xi_\varepsilon \in \partial_X\mathcal{E}_\varepsilon(u_\varepsilon),\ \xi_\varepsilon \rightharpoonup \xi \text{ in } X^* \end{aligned}\right\} \Rightarrow e_0 = \mathcal{E}_0(u) \text{ and } \xi \in \partial_X\mathcal{E}_0(u) \tag{3.1.31}$$

is in general not sufficient to conclude $\xi(t) \in \partial_X\mathcal{E}_0(u(t))$ for a.e. $t \in [0,T]$ since we only have weak convergence of ξ_ε in $\mathrm{L}^2(0,T;X^*)$. Hence, we need the stronger assumption (3.1.23) in Theorem 3.1.6. However, if we additionally assume that $\partial_X\mathcal{E}_0(u) \subset X^*$ is convex (e.g. if $\partial_X\mathcal{E}_0$ is the Fréchet-subdifferential or actually single-valued) it is indeed sufficient to impose (3.1.31).

Proposition 3.1.7. *Assume that for each $u \in X$ the subdifferential $\partial_X\mathcal{E}_0(u)$ is convex. Then, the strong-weak closedness of the graph of $\partial_X\mathcal{E}_\varepsilon$ in* (3.1.31) *implies* (3.1.23).

Proof. Let ξ_ε converge weakly in $\mathrm{L}^2(0,T;X^*)$ to ξ and $\xi_\varepsilon(t) \in \partial_X\mathcal{E}_\varepsilon(u_\varepsilon(t))$ for almost all $t \in [0,T]$. According to [RoS06, Thm. 3.2], there exists a subsequence $\varepsilon_k \to 0$ and a family of Young measures μ_t on X^* (see e.g. [RoS06, Def. 3.1]) such that $\xi(t) = \int_{X^*} \eta\, \mu_t(\mathrm{d}\eta)$ and μ_t is concentrated on the set

$$L(t) = \bigcap_{n=1}^{\infty} \overline{\{\xi_{\varepsilon_k}(t) \,|\, k \geq n\}}^{w} \subset X^*,$$

where the superscript w refers to the weak closure in X^*. Hence, the strong-weak closedness (3.1.31) implies $L(t) \subset \partial_X\mathcal{E}_0(u(t))$ for almost all t and the convexity of $\partial_X\mathcal{E}_0$ yields $\xi(t) \in \partial_X\mathcal{E}_0(u(t))$. $\square$

Finally, let us remark that in the Λ-convex setting of Subsection 3.1.2, condition (3.1.31) and hence also (3.1.23) are always satisfied.

Proposition 3.1.8. *Let $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ be convex, $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in X, and $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$ in X. Then, the Fréchet-subdifferential $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon$ satisfies* (3.1.31).

Proof. The proof follows along the lines of [Mie14, Prop. 2.9] and [Att84, Thm. 3.66]. Due to the quadratic structure of $\mathcal{R}_\varepsilon$ and the convexity of $\mathcal{E}_\varepsilon$ any element $\xi_\varepsilon \in \partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u_\varepsilon)$ satisfies

$$\text{for all } w \in X: \quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u_\varepsilon) + \langle \xi_\varepsilon, w - u_\varepsilon \rangle + \Lambda\mathcal{R}_\varepsilon(w - u_\varepsilon).$$

The strong Γ-convergence of $\mathcal{E}_\varepsilon$ implies: For arbitrarily fixed $\widehat{u} \in X$, there exists a sequence $\widehat{u}_\varepsilon$ such that $\widehat{u}_\varepsilon \to \widehat{u}$ in X and $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(\widehat{u})$. Choosing $w = \widehat{u}_\varepsilon$ and passing to the limit $\varepsilon \to 0$, we obtain $\mathcal{E}_0(\widehat{u}) \geq e_0 + \langle \xi, \widehat{u} - u \rangle + \Lambda\mathcal{R}_0(\widehat{u} - u)$, where we also used that $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$. Setting $\widehat{u} = u$, yields $\mathcal{E}_0(u) \geq e_0$. Finally, we employ the lim inf-estimate for $u_\varepsilon \to u$ in X, which gives $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u)$, and hence we arrive at $e_0 = \mathcal{E}_0(u)$. Altogether, we have shown $\mathcal{E}_0(w) \geq \mathcal{E}_0(u) + \langle \xi, w - u \rangle + \Lambda\mathcal{R}_0(w - u)$ for all $w \in X$ and, therefore, we conclude with $\xi \in \partial_{\mathrm{F}}\mathcal{E}_0(u)$. $\square$

3.2 Homogenization of a Cahn–Hilliard-type equation

In this section, we apply the two approaches established in Section 3.1 to derive homogenization limits of a Cahn–Hilliard-type equation with a microscopic and a macroscopic length scale. In the bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, we consider the fourth order equation written formally as

$$\partial_t u_\varepsilon = \operatorname{div}\big[M_\varepsilon(x)\nabla\big(\partial_u W_\varepsilon(x,u_\varepsilon) - \operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon)\big)\big] \tag{3.2.1}$$

subject to the usual homogeneous Neumann boundary conditions for u and the thermodynamic driving force (also called chemical potential) ξ, namely $A_\varepsilon(x)\nabla u \cdot \nu = 0$ and $M_\varepsilon(x)\nabla\xi \cdot \nu = 0$ with ν denoting the unit outer normal vector to $\partial\Omega$. The multiple scales of the problem are encoded in the periodically oscillating tensors $M_\varepsilon : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $A_\varepsilon : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ as well as the potential $W_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ (see subsequent subsection).

Using Theorem 3.1.5 and Theorem 3.1.6, we show that solutions u_ε of the multiscale Cahn–Hilliard equation (3.2.1) converge in a suitable sense to a solution u of an effective equation that reads

$$\partial_t u = \operatorname{div}\big[M_{\mathrm{eff}}(x)\nabla\big(\partial_u W_{\mathrm{eff}}(x,u) - \operatorname{div}(A_{\mathrm{eff}}(x)\nabla u)\big)\big]. \tag{3.2.2}$$

with M_{eff}, A_{eff}, and W_{eff} being effective (homogenized) quantities, see Propositions 3.2.4 and 3.2.7 in Section 3.2.4 for the precise definition.

3.2.1 Review of related literature

Let us shortly review the literature on E-convergence and homogenization results related to the Cahn–Hilliard equation. An effective macroscopic Cahn–Hilliard equation in a porous media setting is derived in [SP*13] via the method of asymptotic expansion. In [SaS04], energy-based methods, which we term energy-dissipation principle, are developed to derive evolutionary Γ-convergence results for gradient flows in an abstract setting. Based on this, the sharp interface limit of the Cahn–Hilliard equation is investigated in [Le08] using the classical Modica–Mortola energy functional. In [Ser11], the abstract scheme for energies defined on spaces with Hilbert space structure in [SaS04] is generalized to metric spaces. In [BB*12], the convergence of the one dimensional Cahn–Hillard equation to a Stefan problem is proved for nonconvex potentials relying once more on [SaS04]. In [NiO01, NiO10], sharp interface limits are rigorously derived by exploiting the gradient structure of the Cahn–Hilliard equation, Γ-convergence, and the Rayleigh principle. Moreover, the concept of evolutionary Γ-convergence was applied to Hamiltonian systems in [Mie08], and a homogenization result for the wave equation was obtained. In [MRS08] E-convergence of rate-independent systems, which can be seen as generalized gradient systems, was discussed using an energetic formulation which corresponds to the EDP. For a physical application of the homogenization of Cahn–Hilliard-type equations, we refer to [BK*02, TB*03]. Therein, the dewetting process of thin films on heterogeneous substrates is modeled via the Cahn–Hilliard equation with nonlinear mobility and spatially periodic oscillating potential.

3.2.2 Notation and assumptions

In this subsection, we introduce the notation and the assumptions on the given data, that we will use in the subsequent sections to apply the abstract results from Section 3.1. Let us remark that we do not claim that these assumptions are sufficient to prove existence of solutions. In fact, our basic assumption is that solutions of the Cahn–Hilliard equation (3.2.1) always exist (see Definition 3.2.3 for the precise notion of solution). We refer to [ElG96, AbW07, GM*11, Hei15] and the survey article [Nov08] for results in this direction.

Following Section 1.2, we decompose any $x \in \Omega$ into its macroscopic part $\mathcal{N}_\varepsilon(x) = \varepsilon[x/\varepsilon]_Y \in \varepsilon\mathbb{Z}^d$ and its microscopic part $\{x/\varepsilon\}_Y \in \mathcal{Y}$. For notational simplicity, we introduce the (translated) microscopic cell

$$\mathcal{C}_\varepsilon(x) := \mathcal{N}_\varepsilon(x) + \varepsilon Y,$$

where $Y = [0,1)^d$ denotes the unit cell.

We are given two-scale tensors $\mathbb{M} \in \mathrm{L}^\infty(\Omega\times\mathcal{Y};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\mathbb{A} \in \mathrm{L}^\infty(\Omega\times\mathcal{Y};\mathbb{R}^{d\times d}_{\mathrm{sym}})$, which are symmetric and uniformly elliptic with respect to all $(x,y) \in \Omega\times\mathcal{Y}$, i.e.

$$\exists\, \alpha,\ \beta > 0,\ \forall\, \eta \in \mathbb{R}^d : \quad \begin{cases} \alpha|\eta|^2 \leq \eta\cdot\mathbb{M}(x,y)\eta \leq \beta|\eta|^2, \\ \alpha|\eta|^2 \leq \eta\cdot\mathbb{A}(x,y)\eta \leq \beta|\eta|^2. \end{cases} \tag{3.2.3}$$

With $\mathbb{M}$ and $\mathbb{A}$ we then define $M_\varepsilon \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $A_\varepsilon \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ via

$$\begin{gathered} M_\varepsilon(x) := \widehat{M}_\varepsilon(x,\{x/\varepsilon\}_Y) \quad \text{and} \quad A_\varepsilon(x) := \widehat{A}_\varepsilon(x,\{x/\varepsilon\}_Y), \ \text{where} \\ \widehat{M}_\varepsilon(x,y) := \begin{cases} \displaystyle\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{M}(z,y)\,\mathrm{d}z & \text{if } x \in \Omega_\varepsilon^-, \\ \alpha\mathbb{I} & \text{otherwise,} \end{cases} \qquad \text{and} \\ \widehat{A}_\varepsilon(x,y) := \begin{cases} \displaystyle\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{A}(z,y)\,\mathrm{d}z & \text{if } x \in \Omega_\varepsilon^-, \\ \alpha\mathbb{I} & \text{otherwise.} \end{cases} \end{gathered} \tag{3.2.4}$$

Here, $\mathbb{I}$ denotes the identity tensor in $\mathbb{R}^{d\times d}$. Since $\mathbb{M}$ and $\mathbb{A}$ satisfy (3.2.3) for all $(x,y) \in \Omega\times\mathcal{Y}$, it is immediate that M_ε and A_ε satisfy the same estimates in (3.2.3) uniformly with respect to $\varepsilon > 0$ and all $x \in \Omega$. In particular, the extension with $\alpha > 0$ guarantees the uniform ellipticity up to the boundary of Ω.

Finally, for a prescribed two-scale potential $\mathbb{W} : \Omega\times\mathcal{Y}\times\mathbb{R} \to [0,\infty)$ we introduce its macroscopic counterpart $W_\varepsilon : \Omega\times\mathbb{R} \to [0,\infty)$ via

$$W_\varepsilon(x,u) := \widehat{W}_\varepsilon(x,\{x/\varepsilon\}_Y,u) \quad \text{with} \quad \widehat{W}_\varepsilon(x,y,u) := \fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\mathrm{ex}}(z,y,u)\,\mathrm{d}z \quad \forall\, u \in \mathbb{R}, \tag{3.2.5}$$

where for $\mathbb{F} \in \mathrm{L}^1(\Omega\times\mathcal{Y})$ the function $\mathbb{F}_{\mathrm{ex}} \in \mathrm{L}^1(\mathbb{R}^d\times\mathcal{Y})$ denotes the extension by 0 on $(\mathbb{R}^d\backslash\Omega)\times\mathcal{Y}$.

We assume that the potential $\mathbb{W} : \Omega\times\mathcal{Y}\times\mathbb{R} \to [0,\infty)$ is a Carathéodory function, i.e. for all $u \in \mathbb{R}$ the function $(x,y) \mapsto \mathbb{W}(x,y,u)$ is measurable and for a.e. $(x,y) \in \Omega\times\mathcal{Y}$ the function $u \mapsto \mathbb{W}(x,y,u)$ is continuous. Moreover, we make the following simplifying assumptions and refer to Remark 3.2.8 for the more general case of C^1-perturbations of convex potentials. Let $\mathbb{W}$ satisfy uniformly for all $(x,y) \in \Omega \times \mathcal{Y}$

Growth condition:

$$\exists\, C_W \geq 0, \forall\, u \in \mathbb{R}: \quad |\mathbb{W}(x,y,u)| \leq C_W(1+|u|^p), \tag{3.2.6a}$$

where $p < 2^*$ and $2^* \in [1,\infty)$ for $d = 1, 2$ and $2^* = \frac{2d}{d-2}$, for $d \geq 3$;

Uniform modulus of continuity:

$$\exists\, \omega \in \mathrm{C}(\mathbb{R};[0,\infty)) \text{ with } \omega(\bar{u}) \to 0 \text{ for } \bar{u} \to 0, \forall\, u_1, u_2 \in \mathbb{R}: \tag{3.2.6b}$$
$$|\mathbb{W}(x,y,u_1) - \mathbb{W}(x,y,u_2)| \leq \omega(|u_1 - u_2|).$$

Observe that for p as in (3.2.6a), the space $\mathrm{H}^1(\Omega)$ is compactly embedded in $\mathrm{L}^p(\Omega)$. The assumptions (3.2.3)–(3.2.6) suffice to prove the Γ-convergence of the energies $\mathcal{E}_\varepsilon$ in the weak topology of $\mathrm{H}^1(\Omega)$ (see Proposition 3.2.7).

Remark 3.2.1. *Note that the usual ansatz $A_\varepsilon(x) = \mathbb{A}(x, x/\varepsilon)$ for the oscillation coefficients is not well-defined for a general function $\mathbb{A} \in \mathrm{L}^\infty(\Omega\times\mathcal{Y};\mathbb{R}^{d\times d})$ since $\{(x,x/\varepsilon) \in \mathbb{R}^d \times \mathcal{Y}\}$ has null Lebesgue measure. Hence, we are averaging on the microscopic cells $\mathcal{C}_\varepsilon$ with respect to the macroscopic variable x.*

Finally, let us remark that by assuming for all u that $(x,y) \mapsto \mathbb{W}(x,y,u) \in \mathrm{C}(\overline{\Omega}\times\mathcal{Y})$ we can set $W_\varepsilon(x,u) := \mathbb{W}(x,x/\varepsilon,u)$, which would allow us to drop the assumption in (3.2.6b) *and make some of the following proofs more straightforward. However, we want to deal with macroscopic heterostructures and, hence, we consider the more general case here (as in Remark 2.14 in [MiT07]). Here, the definition of M_ε, A_ε in* (3.2.4) *and W_ε in* (3.2.5) *is equivalent to ones in* (2.2.19) *and* (2.2.20) *for the given data of reaction-diffusion systems involving different diffusion length scales.*

3.2.3 Gradient structure of the Cahn–Hilliard equation

The gradient structure of the Cahn–Hilliard equation in (3.2.1) respective (3.2.2) is well-known (cf. [AbW07, Le08, Ser11, BB*12, Hei15]). However, in this section we recall its definition within the abstract framework described in Section 3.1. We allow for $\varepsilon \in [0,1]$ and we identify with $\varepsilon = 0$ the effective quantities M_{eff}, A_{eff}, and W_{eff}.

Obviously, the Cahn–Hilliard equation leaves the average $⨍_\Omega u(t,x)\,\mathrm{d}x$ constant in time. Hence, given an initial value u_0 we set $\varrho := ⨍_\Omega u_0(x)\,\mathrm{d}x$ and define the natural spaces

$$\mathrm{L}^2_\varrho(\Omega) := \{u \in \mathrm{L}^2(\Omega) \,|\, ⨍_\Omega u(x)\,\mathrm{d}x = \varrho\} \quad \text{and} \quad \mathcal{Z}_\varrho := \mathrm{H}^1(\Omega) \cap \mathrm{L}^2_\varrho(\Omega). \tag{3.2.7}$$

The space $\mathcal{Z}_\varrho$ is an affine (and closed) subspace of $\mathrm{H}^1(\Omega)$. On $\mathcal{Z}_\varrho$ the driving functional $\mathcal{E}_\varepsilon : \mathcal{Z}_\varrho \to \mathbb{R}$ is given by the classical Allen–Cahn energy

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \Big[\frac{1}{2}\nabla u \cdot A_\varepsilon(x)\nabla u + W_\varepsilon(x,u)\Big]\,\mathrm{d}x. \tag{3.2.8}$$

We denote the linear space associated with $\mathcal{Z}_\varrho$ by $Z_0 = \mathrm{H}^1(\Omega) \cap \mathrm{L}^2_0(\Omega)$ such that $\mathcal{Z}_\varrho =$

$\varrho + Z_0$. On Z_0 we introduce the (flat) Riemannian structure g_ε via

$$\begin{gathered}\forall\, v_1, v_2 \in Z_0: \quad g_\varepsilon(v_1,v_2) = \int_\Omega \nabla\xi_{v_1}\cdot M_\varepsilon(x)\nabla\xi_{v_2}\,\mathrm{d}x,\\ \text{where } \xi_{v_i}\in \mathrm{H}^1(\Omega) \text{ is the unique solution of } -\operatorname{div}(M_\varepsilon(x)\nabla\xi_{v_i}) = v_i \text{ in } \Omega,\\ \text{satisfying } (M_\varepsilon(x)\nabla\xi_{v_i})\cdot\nu = 0 \text{ on } \partial\Omega \text{ and } \textstyle\fint_\Omega \xi_{v_i}(x)\,\mathrm{d}x = 0.\end{gathered} \tag{3.2.9}$$

Assuming that M_ε is symmetric and positive definite, g_ε clearly defines a scalar product on Z_0. We denote the closure of Z_0 with respect to g with X_0 and easily verify that it is given via

$$X_0 := \left\{ v \in \mathrm{H}^1(\Omega)^* \,|\, \langle v, \mathbb{1}\rangle = 0 \right\}, \tag{3.2.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathrm{H}^1(\Omega)^*$ and $\mathrm{H}^1(\Omega)$ and $\mathbb{1}$ is the constant function with value 1. On X_0 we define the (primal) dissipation potential via

$$\mathcal{R}_\varepsilon(v) := \frac{1}{2} g_\varepsilon(v,v) = \frac{1}{2}\int_\Omega \nabla\xi_v\cdot M_\varepsilon(x)\nabla\xi_v\,\mathrm{d}x, \tag{3.2.11}$$

where $\xi_v \in \mathrm{H}^1(\Omega)$ is defined as in (3.2.9).

The metric tensor g_ε on the tangent space X_0 induces a Riemannian distance on $\mathcal{Z}_\varrho$ which is in our flat case identical to the norm on X_0. The closure of $\mathcal{Z}_\varrho$ with respect to this distance shall be denoted by $\mathcal{X}_\varrho$ and is given via

$$\mathcal{X}_\varrho := \left\{ u \in \mathrm{H}^1(\Omega)^* \,|\, \langle u, \mathbb{1}\rangle = \varrho \right\}. \tag{3.2.12}$$

By the usual embedding of $\mathrm{L}^2(\Omega)$ into $\mathrm{H}^1(\Omega)^*$ we have that Z_0 and $\mathcal{Z}_\varrho$ are densely and compactly embedded in X_0 and $\mathcal{X}_\varrho$, respectively. Moreover, we extend the driving functional $\mathcal{E}_\varepsilon$ to the space $\mathcal{X}_\varrho$ in the usual way by extending it with infinity outside of $\mathcal{Z}_\varrho$.

Let us remark that there are other choices for the space X_0, e.g. by considering $\xi \in \mathrm{H}^1(\Omega)/_{\mathbb{R}}$ and taking $(\mathrm{H}^1(\Omega)/_{\mathbb{R}})^*$ as state space. However, this space is isomorph to X_0.

Proposition 3.2.2. *The space X_0 is isomorph to the space $(\mathrm{H}^1(\Omega)/_{\mathbb{R}})^*$.*

Proof. We construct the isomorphism as follows: By uniquely identifying an equivalence class in $\mathrm{H}^1(\Omega)/_{\mathbb{R}}$ with an element in $\mathrm{H}^1_{\mathrm{av}}(\Omega)$ (meaning $\xi \in \mathrm{H}^1(\Omega)$ and $\fint_\Omega \xi\,\mathrm{d}x = 0$), we can continuously embed the former into the space $\mathrm{H}^1(\Omega)$. We denote this embedding with $I \in \mathrm{Lin}(\mathrm{H}^1(\Omega)/_{\mathbb{R}}; \mathrm{H}^1(\Omega))$.

Moreover, we define $J \in \mathrm{Lin}(\mathrm{H}^1(\Omega); \mathrm{H}^1(\Omega)/_{\mathbb{R}})$ as the linear and continuous map that maps $\xi \in \mathrm{H}^1(\Omega)$ to its equivalence class in $\mathrm{H}^1(\Omega)/_{\mathbb{R}}$. We remark that $\mathrm{ran}(J^*) = X_0$ since J maps $\mathbb{1}$ to 0.

We now claim that $I^* \in \mathrm{Lin}(\mathrm{H}^1(\Omega)^*; (\mathrm{H}^1(\Omega)/_{\mathbb{R}})^*)$ restricted to X_0 is the desired isomorphism whose inverse is given by J^*. For this, let $v \in X_0$ and $\xi \in \mathrm{H}^1(\Omega)$ be given. Denoting by $\langle\cdot,\cdot\rangle_\sim$ the duality product on $(\mathrm{H}^1(\Omega)/_{\mathbb{R}})^*$, we compute

$$\langle (IJ)^* v, \xi\rangle = \langle I^* v, J\xi\rangle_\sim = \langle v, \xi - (\textstyle\fint_\Omega \xi\,\mathrm{d}x)\mathbb{1}\rangle = \langle v, \xi\rangle,$$

where we have used in the last equality that v does not "see" additive constants. Now, let $\tilde v \in (\mathrm{H}^1(\Omega)/_{\mathbb{R}})^*$ and $\tilde\xi \in \mathrm{H}^1(\Omega)/_{\mathbb{R}}$ be given. We easily check that $\langle (JI)^*\tilde v, \tilde\xi\rangle_\sim = \langle J^*\tilde v, I\tilde\xi\rangle = \langle \tilde v, \tilde\xi\rangle_\sim$. Hence, we have shown that $(I^*_{|X_0})^{-1} = J^*$. ☐

As a consequence of Proposition 3.2.2, we identify X_0^* with the space $\mathrm{H}^1(\Omega)/_{\mathbb{R}}$ and consider the dual dissipation potential $\mathcal{R}_\varepsilon^*$ on X_0^*

$$\mathcal{R}_\varepsilon^*(\xi) = \frac{1}{2}\int_\Omega \nabla\xi \cdot M_\varepsilon(x)\nabla\xi \,\mathrm{d}x, \tag{3.2.13}$$

which obviously does not depend on the choice of a representative ξ for an equivalence class in $\mathrm{H}^1(\Omega)/_{\mathbb{R}}$. In particular, we define the map $P_0 : \mathrm{H}^1(\Omega) \to \mathrm{H}^1_{\mathrm{av}}(\Omega)$ via $P_0\xi = \xi - \fint_\Omega \xi \,\mathrm{d}x$, which provides the canonical representative for ξ.

As the metric g_ε depends on $\varepsilon \in [0,1]$ (cf. (3.2.9)), we introduce a topologically equivalent structure on X_0 by associating with $v \in X_0$ the dual variable $\eta \in \mathrm{H}^1(\Omega)$ such that $-\Delta\eta_v = v$, $\nabla\eta_v \cdot \nu = 0$, and $\fint \eta_v \,\mathrm{d}x = 0$. Due to (3.2.3), we have that

$$\forall\, \eta \in \mathrm{H}^1(\Omega) : \quad \frac{\alpha}{2}\int_\Omega |\nabla\eta|^2 \,\mathrm{d}x \le \mathcal{R}_\varepsilon^*(\eta) \le \frac{\beta}{2}\int_\Omega |\nabla\eta|^2 \,\mathrm{d}x.$$

On X_0^* we define the norm $\|\eta\|_{X_0^*} = \|\nabla\eta\|_{\mathrm{L}^2}$, which induces the norm $\|v\|_{X_0} = \|\eta_v\|_{X_0^*}$ on X_0. In particular, we immediately obtain the following uniform estimates for all $\varepsilon \in [0,1]$, cf. (3.1.1),

$$\tfrac{1}{2\beta}\|v\|_{X_0}^2 \le \mathcal{R}_\varepsilon(v) \le \tfrac{1}{2\alpha}\|v\|_{X_0}^2 \quad \text{and} \quad \tfrac{\alpha}{2}\|\xi\|_{X_0^*}^2 \le \mathcal{R}_\varepsilon^*(\xi) \le \tfrac{\beta}{2}\|\xi\|_{X_0^*}^2. \tag{3.2.14}$$

For arbitrary functions $u \in \mathrm{L}^2(0,T;\mathcal{Z}_\varrho)$ with $\dot u \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))^*)$, we have $0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\,\mathrm{d}x = \langle \dot u(t), \mathbb{1}\rangle$, i.e. $\dot u(t) \in X_0$ for almost every $t \in [0,T]$. Therefore, we can consider the projection $P_0(u) = u - \varrho\mathbb{1}$ onto the space $\mathrm{L}^2(0,T;\mathcal{Z}_0) \cap \mathrm{H}^1(0,T;X_0)$. In particular, without loss of generality and for notational consistency with Section 3.1, we set $\varrho = 0$ from now on and consider the function spaces

$$Z := \mathcal{Z}_0 \quad \text{and} \quad X := X_0. \tag{3.2.15}$$

We recall, that for $u \in X$ we denote by $\partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon(u) \subset X^*$ the Fréchet subdifferential of $\mathcal{E}_\varepsilon$ at u with respect to X, which is given via the formula in (3.1.11).

A solution of the Cahn–Hilliard equation is understood in the following sense.

Definition 3.2.3. *Given an initial value $u_0 \in Z$, we call a curve $t \mapsto u(t) \in X$ a solution of the multiscale Cahn–Hilliard equation* (3.2.1), *if it satisfies $0 \in \mathrm{D}\mathcal{R}_\varepsilon(\dot u(t)) + \partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon(u(t))$ in X^* for a.a. $t \in [0,T]$ with $u \in \mathrm{L}^\infty(0,T;Z) \cap \mathrm{H}^1(0,T;X)$ and $u(0) = u_0$.*

3.2.4 Γ-convergence of the energy and dissipation functionals

We use the notion of two-scale convergence (as in Section 1.2) to prove Γ-convergence for the energies and dissipation potentials. Recall that the *periodic unfolding operator* $\mathcal{T}_\varepsilon : \mathrm{L}^q(\Omega) \to \mathrm{L}^q(\mathbb{R}^d \times \mathcal{Y})$, for $1 \le q \le \infty$, is defined via $(\mathcal{T}_\varepsilon u)(x,y) = u_{\mathrm{ex}}(\mathcal{N}_\varepsilon(x) + \varepsilon y)$, where $u_{\mathrm{ex}} \in \mathrm{L}^q(\mathbb{R}^d)$ denotes the extension with 0 outside of Ω.

The Γ-convergence of the dual dissipation potentials $\mathcal{R}_\varepsilon^* : X^* \to [0,\infty)$, cf. (3.2.13), in the weak topology of X^* is well-known. Below, we give a proof based on the periodic unfolding method.

Proposition 3.2.4. *The dual dissipation potentials* $\mathcal{R}_\varepsilon^* : X^* \to [0,\infty)$ Γ*-converge in the weak topology of* X^* *to the limit potential* $\mathcal{R}_0^* : X^* \to [0,\infty)$ *given via*

$$\mathcal{R}_0^*(\xi) = \frac{1}{2}\int_\Omega \nabla\xi \cdot M_{\mathrm{eff}}(x)\nabla\xi \,\mathrm{d}x,$$

where the effective mobility is given via the cell minimization problem

$$\eta \cdot M_{\mathrm{eff}}(x)\eta = \min_{\phi\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})} \fint_{\mathcal{Y}} (\nabla_y\phi + \eta)\cdot \mathbb{M}(x,y)(\nabla_y\phi+\eta)\,\mathrm{d}y. \tag{3.2.16}$$

Proof. The Lipschitz condition for $\partial\Omega$ guarantees $\mathrm{vol}(\{x\in\Omega \,|\, \mathcal{C}_\varepsilon(x) \not\subset \overline{\Omega}\}) \to 0$ as $\varepsilon\to 0$, cf. (1.2.4). With this, Lebesgue's differentiation theorem yields the pointwise convergence

$$(\mathcal{T}_\varepsilon\, M_\varepsilon)(x,y) \to \mathbb{M}_{\mathrm{ex}}(x,y) \text{ for a.a. } (x,y)\in\mathbb{R}^d\times\mathcal{Y}, \tag{3.2.17}$$

cf. Proposition 2.2.6. Thus, the boundedness of $\mathcal{T}_\varepsilon\, M_\varepsilon$ due to (3.2.3) and Lebesgue's dominated convergence theorem yield the strong convergence $\mathcal{T}_\varepsilon\, M_\varepsilon \to \mathbb{M}_{\mathrm{ex}}$ in $\mathrm{L}^q(\mathbb{R}^d\times\mathcal{Y})$ for all $1\le q<\infty$. We now prove the Γ-convergence of $\mathcal{R}_\varepsilon^*$ to $\mathcal{R}_0^*$ in two steps.

1. lim inf-estimate. Let $(\xi_\varepsilon)_\varepsilon \subset X^*$ be a sequence such that $\xi_\varepsilon \rightharpoonup \xi$ in X^*. According to [MiT07, Thm. 2.8], there exists a subsequence (not relabeled) and a function $\Xi \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ such that $\mathcal{T}_\varepsilon\nabla\xi_\varepsilon \rightharpoonup E\nabla\xi + \nabla_y\Xi_{\mathrm{ex}} \in \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. Using the integral identity (1.2.7) and the product rule (1.2.6) in the definition of $\mathcal{R}_\varepsilon^*$, cf. (3.2.13), we obtain

$$\mathcal{R}_\varepsilon^*(\xi_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon\nabla\xi_\varepsilon)\cdot(\mathcal{T}_\varepsilon M_\varepsilon)(x,y)(\mathcal{T}_\varepsilon\nabla\xi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y.$$

With Ioffe's lower semicontinuity result [Iof77] and (3.2.17), we arrive at the lower estimate

$$\liminf_{\varepsilon\to0} \mathcal{R}_\varepsilon^*(\xi_\varepsilon) \ge \frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}} [E\nabla\xi+\nabla_y\Xi_{\mathrm{ex}}]\cdot\mathbb{M}_{\mathrm{ex}}(x,y)[E\nabla\xi+\nabla_y\Xi_{\mathrm{ex}}]\,\mathrm{d}x\,\mathrm{d}y.$$

Finally, we can minimize with respect to the microscopic fluctuations $\nabla_y\Xi$ (see Definition of M_{eff} in (3.2.16)) to get $\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon^*(\xi_\varepsilon) \ge \mathcal{R}_0(\xi)$.

2. Recovery sequence. For given $\widehat{\xi}\in X^*$ and $x\in\Omega$, let $\Phi(x,\cdot)$ denote the unique minimizer for $\eta = \nabla\widehat{\xi}(x)$ in the unit cell problem (3.2.16). In particular, we easily verify that $\Phi\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Exploiting Proposition 2.9 in [MiT07], we can find a sequence $(\widehat{\xi}_\varepsilon)_\varepsilon \subset \mathrm{H}^1_{\mathrm{av}}(\Omega)$ such that $\widehat{\xi}_\varepsilon \rightharpoonup \widehat{\xi}$ in X^* and $\mathcal{T}_\varepsilon\,\nabla\widehat{\xi}_\varepsilon \to E\nabla\widehat{\xi} + \nabla_y\Phi_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. Therefore, with (3.2.17) we arrive at

$$\begin{aligned}\lim_{\varepsilon\to0}\mathcal{R}_\varepsilon^*(\widehat{\xi}_\varepsilon) &= \lim_{\varepsilon\to0}\frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}}(\mathcal{T}_\varepsilon\,\nabla\widehat{\xi}_\varepsilon)\cdot(\mathcal{T}_\varepsilon\, M_\varepsilon)(\mathcal{T}_\varepsilon\,\nabla\widehat{\xi}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}}[E\nabla\widehat{\xi}+\nabla_y\Phi_{\mathrm{ex}}]\cdot\mathbb{M}_{\mathrm{ex}}[E\nabla\widehat{\xi}+\nabla_y\Phi_{\mathrm{ex}}]\,\mathrm{d}x\,\mathrm{d}y = \mathcal{R}_0^*(\widehat{\xi}).\end{aligned}$$

Here, the last identity holds since Φ is a minimizer for minimization problem in the definition of M_{eff}. The lim inf-estimate and the existence of a recovery sequence yield $\mathcal{R}_\varepsilon^* \xrightharpoonup{\Gamma} \mathcal{R}_0^*$ in X^*. $\square$

Remark 3.2.5. *The unique minimizer* $\phi_\eta\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ *of the cell problem* (3.2.16) *solves* $-\mathrm{div}_y(\mathbb{M}(x,y)(\nabla_y\phi_\eta+\eta)) = 0$ *in* $\mathcal{Y}$*. It is called* corrector *as it "corrects" the macroscopic behavior by taking the local fluctuations due to the microscopic structure into account.*

The following result is a direct consequence of the Γ-convergence of $\mathcal{R}_\varepsilon^*$ and the continuity properties of the Legendre transform with respect to Γ-convergence, see Proposition 3.1.4(b).

Corollary 3.2.6. *The primal dissipation potentials $\mathcal{R}_\varepsilon : X \to [0,\infty)$ Γ-converge in the strong topology of X to*

$$v \mapsto \mathcal{R}_0(v) = \mathcal{R}_0^*(\xi_v), \quad \textit{where } -\operatorname{div}(M_{\text{eff}}(x)\nabla\xi_v) = v.$$

The Γ-convergence result for the driving functionals $\mathcal{E}_\varepsilon : Z \to \mathbb{R}$ in (3.2.8) reads as follows.

Proposition 3.2.7. *The family of driving functionals $\mathcal{E}_\varepsilon$ Γ-converges in the weak topology of Z to the limit functional*

$$\mathcal{E}_0(u) = \int_\Omega \Big[\frac{1}{2}\nabla u \cdot A_{\text{eff}}(x)\nabla u + W_{\text{eff}}(x,u)\Big]\,\mathrm{d}x,$$

where the effective quantities are given via

$$\eta \cdot A_{\text{eff}}(x)\eta = \min_{\phi\in \mathrm{H}^1_{\text{av}}(\mathcal{Y})} ⨍_{\mathcal{Y}} (\nabla_y\phi + \eta)\cdot \mathbb{A}(x,y)(\nabla_y\phi+\eta)\,\mathrm{d}y \quad \textit{and}$$
$$W_{\text{eff}}(x,u) = ⨍_{\mathcal{Y}} \mathbb{W}(x,y,u)\,\mathrm{d}y.$$

Proof. For each $\varepsilon\in[0,1]$, we split the family of energy functionals into $\mathcal{E}_\varepsilon = \mathcal{F}_\varepsilon + \mathcal{W}_\varepsilon$, where

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_\Omega \nabla u\cdot A_\varepsilon(x)\nabla u\,\mathrm{d}x \quad \text{and} \quad \mathcal{W}_\varepsilon(u) = \int_\Omega W_\varepsilon(x,u)\,\mathrm{d}x.$$

Here, we write A_0 and W_0 for A_{eff} and W_{eff}, respectively. The convergence $\mathcal{F}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{F}_0$ in Z can be shown analogously to that of the dual dissipation potentials in Proposition 3.2.4. It remains to prove the convergence of the lower order term $\mathcal{W}_\varepsilon(u_\varepsilon)\to\mathcal{W}_0(u)$ for arbitrary sequences $u_\varepsilon \rightharpoonup u$ in Z. Let $(u_\varepsilon)_\varepsilon \subset Z$ be such a sequence and define $U_\varepsilon = \mathcal{T}_\varepsilon u_\varepsilon$. Since Z embeds compactly into $\mathrm{L}^p_0(\Omega)$ for $p<2^*$ as in (3.2.6a), we have $u_\varepsilon \to u$ in $\mathrm{L}^p(\Omega)$ as well as $U_\varepsilon \to Eu$ in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$, cf. (1.2.10) and Proposition 1.2.4(d). Thus, there exists a subsequence (not relabeled) such that $U_\varepsilon(x,y)\to Eu(x,y)$ pointwise for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$. Therefore, exploiting the modulus of continuity in assumption (3.2.6b) gives for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$ the convergence

$$⨍_{\mathcal{C}_\varepsilon(x)} \big|\mathbb{W}_{\text{ex}}(z,y,U_\varepsilon(x,y)) - \mathbb{W}_{\text{ex}}(z,y,Eu(x,y))\big|\,\mathrm{d}z$$
$$\le \omega\big(|U_\varepsilon(x,y)-Eu(x,y)|\big)\to 0. \quad (3.2.18)$$

Moreover, Lebesgue's differentiation theorem yields for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$

$$\lim_{\varepsilon\to0} ⨍_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\text{ex}}(z,y,Eu(x,y))\,\mathrm{d}z = \mathbb{W}_{\text{ex}}(x,y,Eu(x,y)). \quad (3.2.19)$$

Using the integral identity (1.2.7) and the definition of W_ε in (3.2.5) (see also [MiT07, Eq. (2.16)]), we have

$$\int_\Omega W_\varepsilon(x, u_\varepsilon(x))\,\mathrm{d}x = \int_{\mathbb{R}^d\times\mathcal{Y}} \fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\mathrm{ex}}(z, y, U_\varepsilon(x,y))\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

We write

$$\mathcal{W}_\varepsilon(u_\varepsilon) = \mathcal{W}_0(u) + I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon,$$

where

$$\begin{aligned}
I_1^\varepsilon &= \int_{(\mathbb{R}^d\setminus\Omega)\times\mathcal{Y}} \fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\mathrm{ex}}(z, y, U_\varepsilon)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y,\\
I_2^\varepsilon &= \int_{\Omega\times\mathcal{Y}} \fint_{\mathcal{C}_\varepsilon(x)} \left[\mathbb{W}_{\mathrm{ex}}(z, y, U_\varepsilon) - \mathbb{W}_{\mathrm{ex}}(z, y, Eu)\right]\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y,\\
I_3^\varepsilon &= \int_{\Omega\times\mathcal{Y}} \fint_{\mathcal{C}_\varepsilon(x)} \left[\mathbb{W}_{\mathrm{ex}}(z, y, Eu) - \mathbb{W}_{\mathrm{ex}}(x, y, Eu)\right]\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

For $|I_1^\varepsilon| \to 0$, we note that due to the extension by 0 the integrand vanishes everywhere except for a set that is contained in $\mathcal{B}_\varepsilon = (\overline{\Omega_\varepsilon^+} \setminus \Omega_\varepsilon^-) \times \mathcal{Y}$. Since Ω has a Lipschitz boundary the measure of this set tends to 0 and we conclude

$$|I_1^\varepsilon| \le \int_{\mathcal{B}_\varepsilon} C_W(1 + |U_\varepsilon(x,y)|^p)\,\mathrm{d}x\,\mathrm{d}y \to 0.$$

For $|I_2^\varepsilon| + |I_3^\varepsilon| \to 0$, we exploit the pointwise convergence in (3.2.18) and (3.2.19), respectively, as well as Lebesgue's dominated convergence theorem with the same integrable (strongly in $\mathrm{L}^1(\mathbb{R}^d{\times}\mathcal{Y})$ converging) majorant $C_W(1 + |U_\varepsilon(x,y)|^p)$. □

Remark 3.2.8. *For simplicity, we restricted ourselves to potentials $\mathbb{W}$ that satisfy the growth condition in* (3.2.6a)*. However, it is not hard to verify that the Γ-convergence also holds for a bigger class of functionals. In particular, we can relax the growth condition and consider perturbations of convex potentials in the following sense. Let $\mathbb{W}$ admit the decomposition $\mathbb{W} = \mathbb{W}_{\mathrm{cvx}} + \mathbb{W}_{\mathrm{reg}}$, such that $\mathbb{W}$ is bounded from below, $u \mapsto \mathbb{W}_{\mathrm{cvx}}(x, y, u)$ is convex and $u \mapsto \mathbb{W}_{\mathrm{reg}}(x, y, u)$ satisfies the growth condition in* (3.2.6a)*. Additionally, we assume that $\mathbb{W}_{\mathrm{cvx}}$ and $\mathbb{W}_{reg}$ fulfill the modulus of continuity condition* (3.2.6b) *on their domain uniformly with respect to a.a. $(x, y) \in \Omega \times \mathcal{Y}$.*

We immediately check that the lim inf*-estimate follows from Lebesgue's differentiation theorem, condition* (3.2.6b)*, and Fatou's lemma. However, the proof of the* lim sup*-estimate is not so straightforward. The crucial point is that the recovery sequence $(\widehat{u}_\varepsilon)_\varepsilon$ for given $\widehat{u} \in Z$ has to be constructed such that its gradients exhibit the "right" oscillations. We follow the construction given in [MiT07, Prop. 2.9] and set*

$$\widehat{u}_\varepsilon(x) = \widehat{u}(x) + \varepsilon U(t_\varepsilon, x, \tfrac{x}{\varepsilon}), \tag{3.2.20}$$

where $U(t, x, y) = \int_{\mathbb{R}^d}\int_{\mathcal{Y}} K(t, x{-}\tilde{x}, y{-}\tilde{y})\widehat{U}_{ex}(\tilde{x}, \tilde{y})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}$. Here, K is the heat kernel on $\mathbb{R}^d \times \mathcal{Y}$, $\widehat{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{av}(\mathcal{Y}))$ is the solution of the cell problem for $\eta = \nabla\widehat{u}$ in (3.2.18)*, and $t_\varepsilon \to 0$ for $\varepsilon \to 0$. Using Jensen's inequality and a suitable majorant, which we can always assume to exist, we can pass to the limit and obtain the upper estimate.*

However, in the case that the domain of $\mathbb{W}_{\rm cvx}$ *is bounded with respect to* u *we have to guarantee that the recovery sequence is also constrained to the domain. In the case that the domain does not depend on* $(x,y) \in \Omega \times \mathcal{Y}$*, i.e.* $\overline{\operatorname{dom}(\mathbb{W}_{\rm cvx}(x,y,\cdot))} = [a,b]$*, we set* $\widehat{u}_\varepsilon(x) = \delta_\varepsilon(\widehat{u}(x) - m_\varepsilon) + \varepsilon U(t_\varepsilon, x, x/\varepsilon)$*, choose* $t_\varepsilon \to 0$*,* $\delta_\varepsilon \to 1$*, and* $m_\varepsilon \to 0$ *accordingly to get* $a < u_\varepsilon(x) < b$ *for a.a.* $x \in \Omega$*.*

3.2.5 Convergence result based on EVE

In this subsection, we prove the evolutionary Γ-convergence of the Cahn–Hilliard gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to the effective system $(X, \mathcal{E}_0, \mathcal{R}_0)$ by relying on the convexity of $\mathcal{E}_\varepsilon$ with respect to $\mathcal{R}_\varepsilon$. In particular, the key assumption is

$$\exists\, \lambda \in \mathbb{R}\ \forall\, (x,y) \in \Omega \times \mathcal{Y}: \quad u \mapsto \mathbb{W}(x,y,u) - \frac{\lambda}{2}|u|^2 \quad \text{is convex.} \tag{3.2.21}$$

The next lemma shows that the λ-convexity of $\mathbb{W}$ implies Λ-convexity of the driving functionals $\mathcal{E}_\varepsilon$ with respect to $\mathcal{R}_\varepsilon$.

Lemma 3.2.9. *Let* (3.2.21) *be satisfied, then there exists* $\Lambda \in \mathbb{R}$ *such that* $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda \mathcal{R}_\varepsilon(u)$ *is convex.*

Proof. In this proof, we abbreviate $\mathrm{L}^2(\Omega)$ with L^2. It is easy to see that (3.2.21) yields the convexity of $u \mapsto \mathcal{E}_\varepsilon(u) - \frac{\lambda}{2}\|u\|^2_{\mathrm{L}^2} - \frac{\alpha}{2}\|\nabla u\|^2_{\mathrm{L}^2}$ with $\alpha > 0$ from (3.2.3). Namely, for $\theta \in [0,1]$ and $u_0, u_1 \in Z$ we have

$$\mathcal{E}_\varepsilon(u_\theta) \le (1-\theta)\mathcal{E}_\varepsilon(u_0) + \theta \mathcal{E}_\varepsilon(u_1) - \frac{\theta(1-\theta)}{2}\Big(\alpha\|\nabla(u_0 - u_1)\|^2_{\mathrm{L}^2} + \lambda \|u_0 - u_1\|^2_{\mathrm{L}^2}\Big),$$

where $u_\theta = (1-\theta)u_0 + \theta u_1$. Hence, it remains to show that we can find a constant $\Lambda \in \mathbb{R}$ such that the estimate $\Lambda \mathcal{R}_\varepsilon(v) \le \alpha \|\nabla v\|^2_{\mathrm{L}^2} + \lambda \|v\|^2_{\mathrm{L}^2}$ is satisfied for all $v \in Z$. Indeed, due to the embedding $Z \subset \mathrm{L}^2_0(\Omega) \subset X$ and Cauchy's estimate we obtain

$$\forall\, \delta > 0: \quad \|v\|^2_{\mathrm{L}^2} \le \delta \|\nabla v\|^2_{\mathrm{L}^2} + C_\delta \|v\|^2_X.$$

Here, we used Poincaré's inequality, i.e. $\|v\|_{\mathrm{L}^2} \le C_{\mathrm{P}} \|\nabla v\|^2_{\mathrm{L}^2}$ for all $v \in Z$.

Hence, in the case $\lambda = -\lambda_- < 0$ we fix $0 < \delta < \alpha/(\lambda_-)$ and choose $\Lambda \in \mathbb{R}$ such that $\Lambda \le -\lambda_- C_\delta/\alpha$, whereas for $\lambda \ge 0$ we simply set $\Lambda = 0$. With (3.2.14) it is now easy to see that $\mathcal{E}_\varepsilon - \Lambda \mathcal{R}_\varepsilon$ is convex. □

We can now state the first homogenization result, namely the E-convergence of the multiscale Cahn–Hilliard system in the λ-convex case.

Theorem 3.2.10. *Let* $\mathcal{E}_\varepsilon$ *and* $\mathcal{R}_\varepsilon$ *be as before and let* $u_\varepsilon(0) \to u(0)$ *in* X*. Under the additional convexity assumption* (3.2.21) *the solutions* u_ε *of* (3.2.1) *weakly converge in* Z *for each* $t \in [0,T]$*,* $T > 0$*, to the unique solution of the effective Cahn–Hilliard equation* (3.2.2)*. Moreover, for each* $t \in (0,T]$ *the energies converge, i.e.* $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$*.*

Proof. We aim to apply Theorem 3.1.5. For this it remains to show that $\mathcal{R}_\varepsilon(v_\varepsilon) \to \mathcal{R}_0(v)$ for $v_\varepsilon \to v$ strongly in X. Indeed, let a sequence $v_\varepsilon \to v$ strongly in X be given. Moreover, let $\xi_\varepsilon \in X^*$ be the sequence associated with v_ε via solving $-\operatorname{div}(M_\varepsilon \nabla \xi_\varepsilon) = v_\varepsilon$. By standard estimates, we obtain $\xi_\varepsilon \rightharpoonup \xi$ in X^* with ξ such that $-\operatorname{div}(M_{\text{eff}}\nabla\xi) = v$ as in (3.2.9). Thus, we arrive at

$$\lim_{\varepsilon\to 0} \mathcal{R}_\varepsilon(v_\varepsilon) = \frac{1}{2}\lim_{\varepsilon\to 0}\langle v_\varepsilon, \xi_\varepsilon\rangle = \frac{1}{2}\langle v, \xi\rangle = \frac{1}{2}\int_\Omega \nabla\xi \cdot M_{\text{eff}}\nabla\xi \,\mathrm{d}x = \mathcal{R}_0(v),$$

where we have used the strong-weak convergence in the duality product. □

3.2.6 Convergence results based on EDP

In this subsection, we prove the E-convergence of the multiscale system $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ using the energy-dissipation principle EDP discussed in Subsection 3.1.3. In contrast to the previous subsection we drop the λ-convexity of the potential $\mathbb{W}$. Thus, it is in general not clear whether the chain rule in (3.1.4) holds and we have to additionally assume it to be satisfied here.

Regardless of the convexity properties of the energy $\mathcal{E}_\varepsilon$, the EDP formulation requires in any case the well-preparedness of the initial conditions, viz. $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon(0)) = \mathcal{E}_0(u(0)) < \infty$. Moreover, the application of Theorem 3.1.6 rests upon the closedness of the subdifferential $\partial_X\mathcal{E}_\varepsilon$ in the sense of (3.1.23). In the following two propositions, we provide sufficient conditions on the potential $\mathbb{W}$ that guarantee the closedness. In the first proposition, we assume that the potential $\mathbb{W}$ is λ-convex as in (3.2.21).

Proposition 3.2.11. *Assume that the potential* $\mathbb{W}$ *is* λ*-convex as in* (3.2.21)*, then the closedness of the subdifferential* (3.1.23) *holds.*

Proof. It is shown in Lemma 3.2.9 and Theorem 3.2.10 that $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ is convex and $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$ in X, respectively. Thus, the Propositions 3.1.7 and 3.1.8 yield the closedness (3.1.23). □

In the second proposition we replace the convexity assumption with a growth and continuity condition for the derivative of $\mathbb{W}$. In particular, in this case the energies are Fréchet differentiable on $\mathrm{H}^1(\Omega)$ with $\mathrm{D}\mathcal{E}_\varepsilon(u) = -\operatorname{div}(A_\varepsilon(x)\nabla u) + \partial_u W_\varepsilon(x, u)$. Moreover, the growth condition on $\partial_u\mathbb{W}$ implies that for $\mathbb{W}$ in (3.2.6a) with the same exponent. We recall that $P_0 : \mathrm{L}^1(\Omega) \to \mathrm{L}^1_0(\Omega)$ denotes the canonical projection with $P_0(\varphi) = \varphi - \fint_\Omega \varphi \,\mathrm{d}x$.

Proposition 3.2.12. *Assume that* $\mathbb{W} : \Omega\times\mathcal{Y}\times\mathbb{R} \to \mathbb{R}$ *satisfies* $\mathbb{W}(x, y, \cdot) \in \mathrm{C}^1(\mathbb{R})$ *for all* $(x, y) \in \Omega\times\mathcal{Y}$ *as well as*

$$\begin{aligned}
&\text{Growth condition:}\\
&\exists\, C \geq 0,\ \forall\, u \in \mathbb{R}: \quad \big|\partial_u\mathbb{W}(x, y, u)\big| \leq C(1+|u|^{p-1}),\\
&\textit{where } p < 2^* \textit{ and } 2^* \in [1, \infty) \textit{ for } d = 1, 2 \textit{ and } 2^* = \tfrac{2d}{d-2},\ \textit{for } d \geq 3;\\
&\text{Uniform modulus of continuity:}\\
&\exists\, \widehat{\omega} \in \mathrm{C}(\mathbb{R}; [0, \infty)) \textit{ with } \widehat{\omega}(\bar{u}) \to 0 \textit{ for } \bar{u} \to 0,\ \forall\, u_1, u_2 \in \mathbb{R}:\\
&|\partial_u\mathbb{W}(x, y, u_1) - \partial_u\mathbb{W}(x, y, u_2)| \leq \widehat{\omega}(|u_1 - u_2|).
\end{aligned} \tag{3.2.22}$$

Then, $\mathcal{E}_\varepsilon$ is Fréchet differentiable on $\mathrm{H}^1(\Omega)$ for all $\varepsilon \in [0,1]$ with $\mathrm{D}\mathcal{E}_\varepsilon$ denoting the differential. The Fréchet subdifferential of $\mathcal{E}_\varepsilon$ with respect to X is given via

$$\partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon(u) = \begin{cases} \{P_0\left(\mathrm{D}\mathcal{E}_\varepsilon(u)\right)\} & \text{if } \mathrm{D}\mathcal{E}_\varepsilon(u) \in \mathrm{H}^1(\Omega), \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.2.23}$$

Moreover, $\partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon$ satisfies the closedness condition in (3.1.23).

Proof. The Fréchet differentiability on $\mathrm{H}^1(\Omega)$ follows directly from the compact embedding $\mathrm{H}^1(\Omega) \subset \mathrm{L}^p(\Omega)$ and the continuity of the associated Nemytskii operator (for fixed ε)

$$\mathcal{N}_\varepsilon : \begin{cases} \mathrm{L}^p(\Omega) & \to & \mathrm{L}^{p'}(\Omega), \\ u & \mapsto & \partial_u W_\varepsilon(\cdot, u(\cdot)), \end{cases}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The characterization of the subdifferential follows immediately.

It remains to verify the closedness of the Fréchet subdifferential $\partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon$. Since $\partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon$ is convex it is sufficient to prove the strong-weak closedness in X as in (3.1.31) according to Proposition 3.1.7. Hence, let us consider sequences $u_\varepsilon \to u$ in X and $\xi_\varepsilon \rightharpoonup \xi$ in X^* satisfying $\mathcal{E}_\varepsilon(u_\varepsilon) \to e_0$ and $\xi_\varepsilon \in \partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon(u_\varepsilon)$. We follow the lines of the proof of Proposition 3.2.7. Since the energies are uniformly bounded, we can extract a (non-relabeled) subsequence such that $u_\varepsilon \rightharpoonup u$ in Z and $u_\varepsilon \to u$ in $\mathrm{L}^p(\Omega)$ as well as $\mathcal{T}_\varepsilon \nabla u_\varepsilon \rightharpoonup E\nabla u + \nabla_y U_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ with $U \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Moreover, u_ε converges to u almost everywhere in Ω.

We consider a sequence $v_\varepsilon \rightharpoonup v$ in Z, which additionally satisfies the strong convergence $\mathcal{T}_\varepsilon \nabla v_\varepsilon \to E\nabla v + \nabla_y V_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R} \times \mathcal{Y})$, where $V \in \mathrm{L}(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ is arbitrary but fixed. Let us abbreviate $\xi_\varepsilon^W(x) = \partial_u W_\varepsilon(x, u_\varepsilon(x))$. Due to the assumptions in (3.2.22), we can argue as in the proof of Proposition 3.2.7 to deduce $\lim_{\varepsilon\to 0} \int_\Omega \xi_\varepsilon^W v_\varepsilon \, \mathrm{d}x = \int_\Omega \xi_{\mathrm{eff}}^W v \, \mathrm{d}x$, where $\xi_{\mathrm{eff}}^W(x) = \partial_u W_{\mathrm{eff}}(x, u(x))$. Moreover, using the integral identity for the unfolding operator we obtain

$$\langle \xi_\varepsilon, v_\varepsilon \rangle = \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathcal{T}_\varepsilon \nabla v_\varepsilon) \cdot (\mathcal{T}_\varepsilon A_\varepsilon)(\mathcal{T}_\varepsilon \nabla u_\varepsilon) \, \mathrm{d}x \, \mathrm{d}y + \langle \xi_\varepsilon^W, v_\varepsilon \rangle. \tag{3.2.24}$$

Passing to the limit $\varepsilon \to 0$ in (3.2.24) yields

$$\langle \xi, v \rangle = \int_{\Omega \times \mathcal{Y}} [E\nabla v + \nabla_y V] \cdot \mathbb{A}[E\nabla u + \nabla_y U] \, \mathrm{d}x \, \mathrm{d}y + \langle \xi_{\mathrm{eff}}^W, v \rangle, \tag{3.2.25}$$

where we have used $v_\varepsilon \to v$ in X due to the compact embedding $Z \subset X$. We point out that v and V are arbitrary test functions in (3.2.25). On the one hand, we can set $v \equiv 0$ which gives $\int_{\Omega \times \mathcal{Y}} \nabla_y V \cdot \mathbb{A}[\nabla u + \nabla_y U] \, \mathrm{d}x \, \mathrm{d}y = 0$ for all $V \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Thus, U is the unique corrector function associated with u. Indeed, U solves the local problem $-\operatorname{div}_y(\mathbb{A}(x,y)[\nabla u + \nabla_y U]) = 0$ in $\mathcal{Y}$ for a.e. $x \in \Omega$. On the other hand, setting $V \equiv 0$ yields for all $v \in Z$

$$\langle \xi, v \rangle = \int_{\Omega \times \mathcal{Y}} \nabla v \cdot \mathbb{A}[\nabla u + \nabla_y U] + \partial_u \mathbb{W}(u) v \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \nabla v \cdot A_{\mathrm{eff}} \nabla u + \partial_u W_{\mathrm{eff}}(u) v \, \mathrm{d}x.$$

Thus, we conclude that $\xi = \mathrm{D}\mathcal{E}_0(u)$ and $\xi \in \partial_{\mathrm{F}}^X \mathcal{E}_0(u)$.

Finally, it remains to show $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}_0(u)$. For this, it suffices to prove the strong convergence $\mathcal{T}_\varepsilon \nabla u_\varepsilon \to E\nabla u + \nabla_y U_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$. Indeed, using the uniform ellipticity of $\mathcal{T}_\varepsilon A_\varepsilon$ and (3.2.24) gives for $\Xi_\varepsilon = \mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ and $\Xi = E\nabla u + \nabla_y U_{\mathrm{ex}}$

$$\begin{aligned}\alpha\|\Xi_\varepsilon - \Xi\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} &\leq \int_{\mathbb{R}^d\times\mathcal{Y}} (\Xi_\varepsilon{-}\Xi)\cdot \mathcal{T}_\varepsilon A_\varepsilon(\Xi_\varepsilon{-}\Xi)\,\mathrm{d}x\,\mathrm{d}y \\ &= \langle \xi_\varepsilon{-}\xi_\varepsilon^W, u_\varepsilon\rangle - \int_{\mathbb{R}^d\times\mathcal{Y}} \big[2\Xi_\varepsilon\cdot(\mathcal{T}_\varepsilon A_\varepsilon)\Xi - \Xi\cdot(\mathcal{T}_\varepsilon A_\varepsilon)\Xi\big]\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Now, as the right-hand side vanishes for $\varepsilon \to 0$ using (3.2.25), we obtain the strong convergence $\Xi_\varepsilon \to \Xi$ in $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. □

Having collected all sufficient assumptions, we are now in the position to apply Theorem 3.1.6 to the homogenization of the Cahn–Hilliard equation. In particular, the assumptions $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ and $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ in X are satisfied according to the Propositions 3.2.7 and 3.2.4.

Theorem 3.2.13. *Let $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ be as before. We assume that $u_\varepsilon(0) \to u(0)$ in X, the well-preparedness of the initial conditions, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0)) < \infty$, the closedness condition* (3.1.23)*, and the chain rule condition* (3.1.4) *are satisfied. Then, the solutions u_ε of* (3.2.1) *weakly converge in Z for each $t \in [0,T]$, $T > 0$, to a solution u of the effective Cahn–Hilliard equation* (3.2.2)*. Moreover, we have $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$ for each $t \in [0,T]$.*

We complete this subsection by commenting on the well-preparedness condition.

Remark 3.2.14 (Choice of the initial conditions)**.** *The well-preparedness* (3.1.22) *in Theorem 3.2.13 is satisfied for the following choice of initial values. For given $u(0) \in Z$, let $u_\varepsilon(0) \in Z$ be the unique solution of the elliptic problem*

$$\textit{find } \widehat{u} \in Z: \quad \operatorname{div}\big(A_\varepsilon(x)\nabla\widehat{u}\big) = \operatorname{div}\big(A_{\mathrm{eff}}(x)\nabla u(0)\big) \ \textit{in } \Omega, \quad \big(A_\varepsilon(x)\nabla\widehat{u}\big)\cdot\nu = 0 \ \textit{on } \partial\Omega.$$

Then, standard results in periodic homogenization yield $u_\varepsilon(0) \rightharpoonup u(0)$ in Z as well as $\int_\Omega \frac{1}{2}\nabla u_\varepsilon(0)\cdot A_\varepsilon \nabla u_\varepsilon(0)\,\mathrm{d}x \to \int_\Omega \frac{1}{2}\nabla u(0)\cdot A_{\mathrm{eff}}\nabla u(0)\,\mathrm{d}x$, see e.g. [All92]. Employing the compact embedding $Z \subset \mathrm{L}^p_0(\Omega)$ and treating the nonlinearity $\mathbb{W}$ as in Proposition 3.2.7, gives the desired convergence of the initial energies $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$.

In contrast, in the EVE formulation in Theorem 3.2.10, the choice of constant initial values $u_\varepsilon(0) \equiv u_0$ is admissible, since it is not necessary to "recover" the microstructure at $t = 0$. Nevertheless, the convergence of the energies follows for all later times $t > 0$.

3.2.7 Exemplary potentials

We collect three generic potentials as examples which are covered by our theory.

1. We consider the classical *double-well potential*

$$W_{\mathrm{dw}}(u) = \tfrac{1}{4}(u^2{-}1)^2, \tag{3.2.26}$$

which satisfies the growth estimates in (3.2.6) and (3.2.22) for the dimensions $d = 1,2,3$ (see also [ElS86, Ell89]). Moreover, W_{dw} is λ-convex for all $\lambda \leq -1$.

To include different spatial scales in the potential we can consider two-scale functions $\Phi_1, \Phi_2 \in \mathrm{L}^\infty(\Omega\times\mathcal{Y})$ and set $\mathbb{W}_\Phi(x,y,u) = \Phi_1(x,y)W_{\mathrm{dw}}(u) + \Phi_2(x,y)$, which also satisfies the assumptions (3.2.21)–(3.2.22). Moreover, for $\theta \in \mathrm{L}^\infty(\mathcal{Y})$ with $\theta \geq 0$, our multiscale analysis allows us to consider the variant

$$\mathbb{W}_\theta(y,u) = \tfrac14\big(u^2-\theta(y)\big)^2,$$

where the minima are oscillating, i.e. $u_{\min}(x) = \pm(\theta(x/\varepsilon))^{1/2}$. In the limit $\varepsilon \to 0$ we obtain according to Proposition 3.2.7 the effective potential

$$W_{\mathrm{eff}}(u) = \fint_Y \tfrac14(u^2-\theta(y))^2\,\mathrm{d}y = \tfrac14 u^4 - \tfrac12\theta_{\mathrm{arith}}u^2 + \fint_Y \theta(y)^2\,\mathrm{d}y,$$

where $\theta_{\mathrm{arith}} = \fint_Y \theta(y)\,\mathrm{d}y$ denotes the arithmetic mean and the limiting minima are $u_{\min} = \pm(\theta_{\mathrm{arith}})^{1/2}$. Concluding, the Theorems 3.2.10 and 3.2.13 are applicable for $\mathbb{W}_\Phi$ and $\mathbb{W}_\theta$.

2. Another well-known prototypical example is the *logarithmic potential*, cf. [CaH58, CoE92, AbW07], given via

$$W_{\log}(u) = \begin{cases} (u-a)\log(u-a) + (b-u)\log(b-u) - \frac{\kappa}{2}u^2 & \text{if } u \in [a,b], \\ \infty & \text{else,} \end{cases} \tag{3.2.27}$$

with $a < b$ and $\kappa > 0$. Obviously, $W_{\log}$ is λ-convex for all $\lambda \leq -\kappa$. Hence, the Theorems 3.2.10 and 3.2.13 apply to $W_{\log}$, cf. also Remark 3.2.8. We refer to [AbW07] for a characterization of the single-valued Fréchet subdifferential.

An interesting variation of (3.2.27) is to consider oscillating boundaries $a_\varepsilon(x) = a(x/\varepsilon)$ and $b_\varepsilon(x) = b(x/\varepsilon)$, where $a, b \in \mathrm{L}^\infty(\mathcal{Y})$ are given with $a_{\max} < b_{\min}$. However, it is an open problem to determine the effective limit domain $[a_0, b_0]$ for $\varepsilon \to 0$.

3. As a *nonconvex example* we consider the potential

$$W_\gamma(u) = \tfrac12 u^2 - \tfrac{1}{\gamma+1}|u|^{\gamma+1} \quad \text{with} \quad \gamma \in (\tfrac12, 1). \tag{3.2.28}$$

The function W_γ satisfies the assumptions in (3.2.6) and (3.2.22) with $W_\gamma'(u) = u-|u|^{\gamma-1}u$. Indeed, W_γ' is globally γ-Hölder continuous as we have

$$\forall\, u_0, u_1 \in \mathbb{R}: \quad \big||u_0|^{\gamma-1}u_0 - |u_1|^{\gamma-1}u_1\big| \leq C_\gamma |u_0-u_1|^\gamma,$$

where $C_\gamma = 1$, if $u_0u_1 \geq 0$, and $C_\gamma = 2^{1-\gamma}$, if $u_0u_1 < 0$. The latter follows from the concavity of $u \mapsto |u|^\gamma$ and choosing $\theta = 1/2$ for $u_\theta = (1-\theta)u_0 + \theta(-u_1)$.

However, the function W_γ is clearly not λ-convex since $W_\gamma''(u) = 1-\gamma|u|^{\gamma-1} \to -\infty$ for $|u| \to 0$ and for any $\lambda \in \mathbb{R}$. In particular, there exists no $\Lambda \in \mathbb{R}$ such that $u \mapsto \mathcal{E}(u) - \Lambda\mathcal{R}(u)$ is convex. To see this, we consider an arbitrary $\Lambda \in \mathbb{R}$ and set

$$\mathcal{F}_\Lambda(u) := \mathcal{E}(u) - \Lambda\mathcal{R}(u) = \mathcal{Q}_\Lambda(u) - \int_\Omega \tfrac{1}{\gamma+1}|u|^{\gamma+1}\,\mathrm{d}x,$$
$$\text{where} \quad \mathcal{Q}_\Lambda(u) := \int_\Omega \tfrac12[\nabla u\cdot A\nabla u + u^2]\,\mathrm{d}x - \Lambda\mathcal{R}(u)$$

comprises the quadratic terms. For smooth functions v, the second variation reads

$$\mathrm{D}^2\mathcal{F}_\Lambda(u)[v,v] = 2\mathcal{Q}_\Lambda(v) - \gamma\int_\Omega |u|^{\gamma-1}v^2\,\mathrm{d}x$$

and for each $\Lambda \in \mathbb{R}$ we can find some $u \in Z$ such that $\mathrm{D}^2\mathcal{F}_\Lambda(u)[v,v] < 0$. Hence, the convexity condition (3.1.10) for the EVE formulation is violated and W_γ is a counterexample, for which Theorem 3.2.10 is not applicable.

However, we can still exploit the EDP formulation and apply Theorem 3.2.13 provided we can verify the chain rule (3.1.4). We refer to [RoS06, RSS08] for gradient formulations of non-convex driving functionals and the role of the chain rule. For our particular example, we drop the subscripts and write A for the tensors A_ε and A_{eff}, respectively, and prove the following theorem for $\mathcal{E} \equiv \mathcal{E}_\varepsilon$ with $\varepsilon \in [0,1]$.

Theorem 3.2.15. *Assume that $\partial\Omega$ is of class C^2, $A \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{d\times d}_{\text{spd}})$, and that W_γ is as in* (3.2.28)*. Then, the Fréchet subdifferential (with respect to X) of the energy functional $\mathcal{E} : X \to \mathbb{R}_\infty$ is given by*

$$\partial^X_{\mathrm{F}}\mathcal{E}(u) = \begin{cases} \{-\operatorname{div}(A\nabla u) + P_0 W_\gamma'(u)\} & \text{if } \operatorname{div}(A\nabla u) \in \mathrm{H}^1(\Omega) \text{ and } \\ & (A\nabla u)\cdot\nu = 0 \text{ on } \partial\Omega, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.2.29}$$

Moreover, $\mathcal{E}$ satisfies the chain rule condition (3.1.4)*.*

We conclude that the homogenization result in Theorem 3.2.13 is applicable. To prove Theorem 3.2.15, we use the following integration by parts formula, which is proven in [MeS08].

Theorem 3.2.16 ([MeS08], Thm. 3.1)**.** *Let $\Omega \subset \mathbb{R}^d$ with uniform C^2 boundary $\partial\Omega$ and $A \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})$ be given. Then, for $u \in \mathrm{W}^{2,r}(\Omega)$ with $1 < r < \infty$ we have*

$$\begin{aligned} -(r-1)\int_\Omega |u|^{r-2}\nabla u\cdot A(x)\nabla u\,\mathrm{d}x = \int_\Omega & u|u|^{r-2}\operatorname{div}(A(x)\nabla u)\,\mathrm{d}x \\ & - \int_{\partial\Omega} u|u|^{r-2}\nabla u\cdot A(x)\nu\,\mathrm{d}S_x. \end{aligned} \tag{3.2.30}$$

Proof of Theorem 3.2.15. We prove that the energy functional $\mathcal{E}$ given by $\mathcal{E}(u) = \int_\Omega \frac12\nabla u\cdot A\nabla u + W_\gamma(u)\,\mathrm{d}x$ with $W_\gamma(u) = \frac12 u^2 - \frac{1}{\gamma+1}|u|^{\gamma+1}$ satisfies the following chain rule: If $u \in \mathrm{H}^1(0,T;X)$, $\xi \in \mathrm{L}^2(0,T;X^*)$ such that $\xi(t) \in \partial^X_{\mathrm{F}}\mathcal{E}(u(t))$ for a.a. $t \in [0,T]$, and the function $t \mapsto \mathcal{E}(u(t))$ is bounded, then it is also absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) = \langle \dot u(t), \xi(t)\rangle \quad \text{for a.e. } t \in [0,T]. \tag{3.2.31}$$

The proof follows the basic ideas of [RoS06, Thm. 4], where the sum of a convex functional and a concave perturbation is considered. Thus, we write $W_\gamma = W_1 - W_2$, where $W_1(u) = \frac12 u^2$ and $W_2(u) = \frac{1}{\gamma+1}|u|^{\gamma+1}$. Analogously, we decompose the energy into

$$\begin{gathered} \mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 \text{ on } Z \quad \text{and} \quad \mathcal{E} = +\infty \text{ on } X\backslash Z, \text{ where} \\ \mathcal{E}_1(u) := \int_\Omega \tfrac12\nabla u\cdot A(x)\nabla u + W_1(u)\,\mathrm{d}x \quad \text{and} \quad \mathcal{E}_2(u) := \int_\Omega W_2(u)\,\mathrm{d}x. \end{gathered} \tag{3.2.32}$$

We easily check that $\mathcal{E}$, $\mathcal{E}_1$, and $\mathcal{E}_2$ are Fréchet differentiable on Z. In particular, if $\mathcal{E}$ is Fréchet subdifferentiable in some $u \in X$ we have that

$$\partial^X_{\mathrm{F}}\mathcal{E}(u) = \{-\operatorname{div}(A(x)\nabla u) + P_0 W_\gamma'(u)\} \subset X^* \quad \text{with} \quad (A(x)\nabla u)\cdot\nu = 0 \text{ on } \partial\Omega.$$

Moreover, since $\mathcal{E}_1$ and $\mathcal{E}_2$ are convex, they separately satisfy the chain rule in (3.2.31) according to e.g. [Bré73, Chap. III Lem. 3.3] or [Sho97, Chap. IV Lem. 4.3]. Hence, it remains to prove that $\xi \in \mathrm{L}^2(0,T;X^*)$, satisfying $\xi(t) \in \partial_{\mathrm{F}}^X \mathcal{E}(u(t))$ for a.e. $t \in [0,T]$ with $u \in \mathrm{H}^1(0,T;X)$, can be decomposed into $\xi = \xi_1 - \xi_2$, where $\xi_i \in \mathrm{L}^2(0,T;X^*)$ and $\xi_i(t) \in \partial_{\mathrm{F}}^X \mathcal{E}_i(u(t))$ is satisfied for a.e. $t \in [0,T]$.

First, let us note that the boundedness of $t \mapsto \mathcal{E}(u(t))$ implies $u \in \mathrm{L}^\infty(0,T;Z)$, which in turn means that at least $t \mapsto W_\gamma'(u(t)) = |u(t)|^{\gamma-1}u(t) \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ is satisfied for $\frac{1}{2} < \gamma < 1$.

Due to the smoothness of $\partial\Omega$ and A, we obtain higher regularity of u, namely $u \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))$, see e.g. [Lóp13, Thm. 5.11]. Thus, we can apply Theorem 3.2.16 with $r = 2\gamma \in (1,2)$ to obtain

$$\begin{aligned}\alpha(2\gamma-1)\int_0^T \int_\Omega |u|^{2(\gamma-1)}|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t &\leq \int_0^T \int_\Omega |u|^{2\gamma-1}|\operatorname{div}(A(x)\nabla u)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\left(\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))}\right),\end{aligned}$$

where $\alpha > 0$ is from (3.2.3). Note that the boundary integral in (3.2.30) vanishes since u satisfies $(A(x)\nabla u)\cdot\nu = 0$ on $\partial\Omega$. Since the right-hand side in the above estimate is finite we obtain that $\xi_2 := W_2'(u) = |u|^{\gamma-1}u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$. Thus, we have shown the decomposition and therefore also the chain rule. □

3.3 Conclusion

We conclude our text with a comparison of the approaches for evolutionary Γ-convergence of gradient systems $(X,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ in Section 3.1 based on the evolutionary variational estimate EVE and the energy-dissipation principle EDP.

1. Both abstract results rely on the strong Γ-convergence of the energy functionals $\mathcal{E}_\varepsilon$ in X. Let us remark that we even have Mosco convergence of $\mathcal{E}_\varepsilon$ for the homogenization of the Cahn–Hilliard equation.
2. While the strong Γ-convergence of the dissipation potentials $\mathcal{R}_\varepsilon$ in X is sufficient for EDP, we have to assume additionally continuous convergence in the EVE formulation. The latter is satisfied for the homogenization of Cahn–Hilliard-type equations in Section 3.2.
3. The initial values, which are assumed to converge strongly in X, have to be well-prepared in the EDP case, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$. In particular, this means that $u_\varepsilon(0) \in \mathrm{dom}(\mathcal{E}_\varepsilon)$ has to hold for $\varepsilon \in [0,1]$ for EDP while EVE only requires $u_\varepsilon(0) \in \overline{\mathrm{dom}\,\mathcal{E}_\varepsilon}^X$.
4. The identification of the limit system in the EDP formulation relies on the closedness of the subdifferential $\partial_X\mathcal{E}_\varepsilon$ (see (3.1.23)), which is automatically satisfied for Λ-convex energy functionals.
5. The EVE formulation is based on the convexity of $\mathcal{E}_\varepsilon - \Lambda\mathcal{R}_\varepsilon$, which is always satisfied for λ-convex potentials $\mathbb{W}$ in the Cahn–Hilliard setting, see Lemma 3.2.9. Moreover, the Λ-convexity of $\mathcal{E}_\varepsilon$ implies many desirable properties of the gradient system,

see e.g. [RoS06, DaS10]. In particular, the well-known double-well and logarithmic potentials $W_{\rm dw}$ and $W_{\log}$ fit into this setting. The EDP formulation allows us to consider also energy functionals that are not Λ-convex. In this case, the chain rule condition is not automatically satisfied and its verification may be cumbersome. For instance, the potential W_γ in (3.2.28) is not λ-convex, though the associated energy functional fulfills the chain rule, see Theorem 3.2.15.

Let us remark that our approach is related to [Mie14]. There, Theorem 3.6 gives an abstract E-convergence result based on EDP. Note, however, that more general dissipation potentials are considered, which are also allowed to depend on the state u. However, there it is assumed that the dissipation potentials satisfy $\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon(u_\varepsilon,v_\varepsilon)\geq\mathcal{R}_0(u,v)$ for sequences $u_\varepsilon\to u$ in X and $v_\varepsilon\rightharpoonup v$ in X. For the Cahn–Hilliard dissipation potential this lim inf-estimate is not satisfied: Indeed, for $v_\varepsilon\rightharpoonup v$ in X, we consider

$$\mathcal{R}_\varepsilon(v_\varepsilon)=\int_\Omega\tfrac12\nabla\xi_{v_\varepsilon}\cdot M_\varepsilon(x)\nabla\xi_{v_\varepsilon}\,\mathrm{d}x,\quad\text{where}\quad-\operatorname{div}(M_\varepsilon(x)\nabla\xi_{v_\varepsilon})=v_\varepsilon\text{ as in (3.2.9).}$$

The boundedness of $(v_\varepsilon)_\varepsilon\subset X$ implies the boundedness of $(\xi_{v_\varepsilon})_\varepsilon\subset X^*$ and thus, we obtain $\xi_{v_\varepsilon}\rightharpoonup\xi$ in X^* (up to subsequence). For arbitrary test functions $\varphi_\varepsilon\in X^*$, we study the weak formulation

$$\int_\Omega\nabla\varphi_\varepsilon\cdot M_\varepsilon(x)\nabla\xi_{v_\varepsilon}\,\mathrm{d}x=\langle v_\varepsilon,\varphi_\varepsilon\rangle.\tag{3.3.1}$$

Since M_ε is oscillating and not strongly convergent, the test function φ_ε has to capture the "right oscillations" in order to pass to the limit in the left-hand side. In particular, φ_ε satisfies $\varphi_\varepsilon\rightharpoonup\varphi$ in X^* and $\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon)\to E\nabla\varphi+\nabla_y\Phi_{\rm ex}$ in $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. However, since v_ε is also only weakly converging we cannot pass to the limit in the right-hand side to establish a connection between the limits ξ and v. Thus, from the lower estimate

$$\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon(v_\varepsilon)=\liminf_{\varepsilon\to0}\mathcal{R}^*_\varepsilon(\xi_{v_\varepsilon})\geq\mathcal{R}_0(\xi)$$

we cannot conclude $\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon(v_\varepsilon)\geq\mathcal{R}_0(v)$.

Finally, let us compare our approach to the well-known Sandier & Serfaty result for evolutionary Γ-convergence in [SaS04]. There, also the EDP formulation (Section 3.1.3) is considered in the abstract setting. The crucial conditions can be formulated as

$$\text{i) }\forall\,s\in[0,T):\quad\liminf_{\varepsilon\to0}\int_0^s\mathcal{R}_\varepsilon(v_\varepsilon(s))\,\mathrm{d}s\geq\int_0^s\mathcal{R}_0(v(s))\,\mathrm{d}s$$

$$\text{ii) }\liminf_{\varepsilon\to0}\mathcal{R}^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t)))\geq\mathcal{R}^*_0(-\mathrm{D}\mathcal{E}_0(u(t))).$$

In particular, the conditions are formulated in a very general manner, e.g. the precise notion of the convergence of u_ε and v_ε is not explicitly stated and depends on the concrete problem. In contrast, we provide "easy" to check conditions for $\mathcal{R}_\varepsilon$ and $\mathcal{E}_\varepsilon$. Moreover, we do not need an independent bound for each of the terms $\int_0^T\mathcal{R}_\varepsilon\,\mathrm{d}t$ and $\int_0^T\mathcal{R}^*_\varepsilon\,\mathrm{d}t$.

Remark 3.3.1. *We briefly comment on the connection between our homogenization result for Cahn–Hilliard-type equations and the convergence results for reaction-diffusion systems in Chapter 2. In principle, a generalized version of Gronwall's lemma is applicable to*

Cahn–Hilliard-type equations (3.0.3), *see e.g. [BaM11]. With this, it might be possible to derive explicit rates for the convergence of the solutions by following the approach via Gronwall-type estimates and controlled error terms. Therefore, gradient folding operators for higher derivatives may need to be invented. In contrast, evolutionary* Γ*-convergence does not give rise to quantitative estimates.*

Since we used several times the compact embedding of Z *into* X*, it is not obvious at all whether slow diffusion such as* $\varepsilon^2 M_\varepsilon$ *or* $\varepsilon^2 A_\varepsilon$ *can be treated with the presented theory.*

Bibliography

[ABK09] S. Alonso, M. Bär, and R. Kapral. Effective medium approach for heterogeneous reaction-diffusion media. *J. Chem. Phys.*, 131, 214102, 2009.

[AbW07] H. Abels and M. Wilke. Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy. *Nonlinear Anal.*, 67(11), 3176–3193, 2007.

[ADH90] T. Arbogast, J. Douglas, Jr., and U. Hornung. Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Anal.*, 21(4), 823–836, 1990.

[AGS05] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[All92] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(1), 1482–1518, 1992.

[Att84] H. Attouch. *Variational Convergence for Functions and Operators.* Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[BaM11] S. Bartels and R. Müller. Error control for the approximation of Allen–Cahn and Cahn–Hilliard equations with a logarithmic potential. *Numer. Math.*, 119(3), 409–435, 2011.

[BB*12] G. Bellettini, L. Bertini, M. Mariani, and M. Novaga. Convergence of the one-dimensional Cahn–Hilliard equation. *SIAM J. Math. Anal.*, 44(5), 3458–3480, 2012.

[BeK83] D. Bedeaux and R. Kapral. The effective reaction rate and diffusion coefficients for a two-phase medium. *J. Chem. Phys.*, 79, 1783, 1983.

[Bén72] P. Bénilan. Solutions intégrales d'équations d'évolution dans un espace de Banach. *C. R. Acad. Sci. Paris Sér. A-B*, 274, A47–A50, 1972.

[BK*02] L. Brusch, H. Kühne, U. Thiele, and M. Bär. Dewetting of thin films on heterogeneous substrates: Pinning versus coarsening. *Phys. Rev. E*, 66, 011602, 2002.

[BLP78] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*, volume 5 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 1978.

[BoH03] D. Bothe and D. Hilhorst. A reaction-diffusion system with fast reversible reaction. *J. Math. Anal. Appl.*, 286(1), 125–135, 2003.

[Bra07] D. Braess. *Finite Elemente: Theorie, Schnelle Löser und Anwendungen in der Elastizitätstheorie*. Springer-Verlag Berlin Heidelberg, 4th edition, 2007.

[Bré73] H. Brézis. *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*. North-Holland Publishing Co., Amsterdam, 1973.

[CaH58] J. W. Cahn and J. E. Hilliard. Free energy of a non-uniform system I. Iinterfacial free energy. *J. Chem. Phys.*, 28, 258–267, 1958.

[CDG02] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1), 99–104, 2002.

[CDG08] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM J. Math. Anal.*, 40(4), 1585–1620, 2008.

[CDN10] M. Costabel, M. Dauge, and S. Nicaise. Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains. https://hal.archives-ouvertes.fr/hal-00453934, 2010.

[CiD99] D. Cioranescu and P. Donato. *An Introduction to Homogenization*, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1999.

[ClP99] G. W. Clark and L. A. Packer. Two-scale homogenization of non-linear degenerate evolution equations. *J. Math. Anal. Appl.*, 238(1), 316–328, 1999.

[CoE92] M. I. M. Copetti and C. M. Elliott. Numerical analysis of the Cahn–Hilliard equation with a logarithmic free energy. *Numer. Math.*, 63(1), 39–65, 1992.

[Dam05] A. Damlamian. An elementary introduction to periodic unfolding. *Math. Sci. Appl.*, 24, 119–136, 2005.

[DaS08] S. Daneri and G. Savaré. Eulerian calculus for the displacement convexity in the Wasserstein distance. *SIAM J. Math. Analysis*, 40, 1104–1122, 2008.

[DaS10] S. Daneri and G. Savaré. Lecture notes on gradient flows and optimal transport. arXiv:1009.3737v1, 2010.

[Eck05] C. Eck. Homogenization of a phase field model for binary mixtures. *Multiscale Model. Simul.*, 3(1), 1–27 (electronic), 2004/05.

[EK*10] E. K. Essel, K. Kuliev, G. Kulieva, and L.-E. Persson. Homogenization of quasilinear parabolic problems by the method of Rothe and two scale convergence. *Appl. Math.*, 55(4), 305–327, 2010.

[ElG96] C. M. Elliott and H. Garcke. On the Cahn–Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.*, 27(2), 404–423, 1996.

[Ell89] C. M. Elliott. The Cahn–Hilliard model for the kinetics of phase separation. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 35–73. Birkhäuser, Basel, 1989.

[ElS86] C. M. Elliott and Z. Songmu. On the Cahn–Hilliard equation. *Arch. Rational Mech. Anal.*, 96(4), 339–357, 1986.

[Els02] J. Elstrodt. *Maß- und Integrationstheorie*. Springer-Verlag, Berlin, 3rd edition, 2002.

[Emm04] E. Emmrich. *Gewöhnliche und Operator-Differentialgleichungen. Eine integrierte Einführung in Randwertprobleme und Evolutionsgleichungen für Studierende.* Wiesbaden: Vieweg, 2004.

[Eva98] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[EvG92] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[FA*11] T. Fatima, N. Arab, E. P. Zemskov, and A. Muntean. Homogenization of a reaction-diffusion system modeling sulfate corrosion of concrete in locally periodic perforated domains. *J. Engrg. Math.*, 69(2-3), 261–276, 2011.

[FH*07] L. Flodén, A. Holmbom, M. Olsson, and N. Svanstedt. Reiterated homogenization of monotone parabolic problems. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 53(2), 217–232, 2007.

[FlO06] L. Flodén and M. Olsson. Reiterated homogenization of some linear and nonlinear monotone parabolic operators. *Can. Appl. Math. Q.*, 14(2), 149–183, 2006.

[FlO07] L. Flodén and M. Olsson. Homogenization of some parabolic operators with several time scales. *Appl. Math.*, 52(5), 431–446, 2007.

[FMP12] T. Fatima, A. Muntean, and M. Ptashnyk. Unfolding-based corrector estimates for a reaction-diffusion system predicting concrete corrosion. *Appl. Anal.*, 91(6), 1129–1154, 2012.

[GGZ74] H. Gajewski, K. Gröger, and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Akademie-Verlag, Berlin, 1974. Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, Band 38.

[GiT01] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[GlH97] A. Glitzky and R. Hünlich. Global estimates and asymptotics for electro-reaction-diffusion systems in heterostructures. *Appl. Anal.*, 66(3-4), 205–226, 1997.

[GM*11] M. Grasselli, A. Miranville, R. Rossi, and G. Schimperna. Analysis of the Cahn–Hilliard equation with a chemical potential dependent mobility. *Comm. Partial Differential Equations*, 36(7), 1193–1238, 2011.

[GPS14] I. Graf, M. A. Peter, and J. Sneyd. Homogenization of a nonlinear multiscale model of calcium dynamics in biological cells. *J. Math. Anal. Appl.*, 419(1), 28–47, 2014.

[Gri04] G. Griso. Error estimate and unfolding for periodic homogenization. *Asymptot. Anal.*, 40(3-4), 269–286, 2004.

[Gri05] G. Griso. Interior error estimate for periodic homogenization. *C. R. Math. Acad. Sci. Paris*, 340(3), 251–254, 2005.

[GrP14] I. Graf and M. A. Peter. Homogenization of a carcinogenesis model with different scalings with the homogenization parameter. *Math. Bohem.*, 139(2), 163–184, 2014.

[GrR05] C. Grossmann and H.-G. Roos. *Numerische Behandlung Partieller Differentialgleichungen.* Wiesbaden: Teubner, 3rd revised and expanded ed. edition, 2005.

[HaK12] H. Hanke and D. Knees. Derivation of an effective damage model with evolving micro-structure. *WIAS Preprint No. 1749*, 2012.

[HaK15] H. Hanke and D. Knees. Homogenization of elliptic systems with non-periodic, state dependent coefficients. *Asymptot. Anal.*, 92, 203–234, 2015.

[Han11] H. Hanke. Homogenization in gradient plasticity. *Math. Models Methods Appl. Sci.*, 21, 1651–1684, 2011.

[Hei15] M. Heida. On systems of Cahn–Hilliard and Allen–Cahn equations considered as gradient flows in Hilbert spaces. *J. Math. Anal. Appl.*, 423(1), 410–455, 2015.

[Hen81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1981.

[HJM94] U. Hornung, W. Jäger, and A. Mikelić. Reactive transport through an array of cells with semi-permeable membranes. *RAIRO Modél. Math. Anal. Numér.*, 31, 1257–1285, 1994.

[Hor97] U. Hornung. Introduction. In *Homogenization and porous media*, volume 6 of *Interdiscip. Appl. Math.*, pages 1–25, 259–275. Springer, New York, 1997.

[HSW05] A. Holmbom, N. Svanstedt, and N. Wellander. Multiscale convergence and reiterated homogenization of parabolic problems. *Appl. Math.*, 50(2), 131–151, 2005.

[Iof77] A. D. Ioffe. On lower semicontinuity of integral functionals. I. *SIAM J. Control Optimization*, 15(4), 521–538, 1977.

[JKO94] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of Differential Operators and Integral Functionals.* Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.

[KB*09] Y. Kyrychko, K. Blyuss, S. Hogan, and E. Schöll. Control of spatiotemporal patterns in the gray–scott model. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 19(4), 043126, 2009.

[Kee00] J. P. Keener. Homogenization and propagation in the bistable equation. *Physica D*, 136, 1, 2000.

[Kol04] T. Kolokolnikov. *Pattern Formation in Reaction-Diffusion Models Far from the Turing Regime*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–The University of British Columbia (Canada).

[KWW06] T. Kolokolnikov, M. J. Ward, and J. Wei. Zigzag and breakup instabilities of stripes and rings in the two-dimensional Gray-Scott model. *Stud. Appl. Math.*, 116(1), 35–95, 2006.

[Le08] N. Q. Le. A gamma-convergence approach to the Cahn–Hilliard equation. *Calc. Var. Partial Differential Equations*, 32(4), 499–522, 2008.

[LiR15] M. Liero and S. Reichelt. Homogenization of Cahn–Hilliard-type equations via evolutionary Γ-convergence. *WIAS Preprint No. 2114*, 2015.

[LNW02] D. Lukkassen, G. Nguetseng, and P. Wall. Two-scale convergence. *Int. J. Pure Appl. Math.*, 2, 35–86, 2002.

[Lóp13] J. López-Gómez. *Linear Second Order Elliptic Operators*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[Mah13] H. S. Mahato. *Homogenization of a System of Nonlinear Multi-Species Diffusion-Reaction Equations in an $H^{1,p}$ Setting*. PhD thesis, Universität Bremen, 2013.

[MaK06] V. A. Marchenko and E. Y. Khruslov. *Homogenization of Partial Differential Equations*. Birkhäuser Boston Inc., Boston, MA, 2006. Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky.

[MeM10] S. A. Meier and A. Muntean. A two-scale reaction-diffusion system: homogenization and fast-reaction limits. In *Current advances in nonlinear analysis and related topics*, volume 32 of *GAKUTO Internat. Ser. Math. Sci. Appl.*, pages 443–461. Gakkōtosho, Tokyo, 2010.

[MeS08] G. Metafune and C. Spina. An integration by parts formula in Sobolev spaces. *Mediterr. J. Math.*, 5(3), 357–369, 2008.

[Mie08] A. Mielke. Weak-convergence methods for Hamiltonian multiscale problems. *Discrete Contin. Dyn. Syst.*, 20(1), 53–79, 2008.

[Mie11] A. Mielke. A gradient structure for reaction-diffusion systems and for energy-drift-diffusion systems. *Nonlinearity*, 24, 1329–1346, 2011.

[Mie13] A. Mielke. Thermomechanical modeling of energy-reaction-diffusion systems, including bulk-interface interactions. *Discr. Cont. Dynam. Systems Ser. S*, 6(2), 479–499, 2013.

[Mie14] A. Mielke. On evolutionary Γ-convergence for gradient systems. *WIAS Preprint No. 1915*, 2014. Accepted in Lecture Notes in Appl. Math. Mech.

[Mie15] A. Mielke. Deriving amplitude equations via evolutionary Γ-convergence. *Discr. Cont. Dynam. Systems Ser. A*, 35(6), 2679–2700, 2015.

[MiR13] A. Mielke and E. Rohan. Homogenization of elastic waves in fluid-saturated porous media using the biot model. *Math. Models Methods Appl. Sci.*, 23, 873–916, 2013.

[MiT07] A. Mielke and A. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM J. Math. Anal.*, 39(2), 642–668, 2007.

[MRS90] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. *Semiconductor Equations.* Springer-Verlag, Vienna, 1990.

[MRS08] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Partial Differential Equations*, 31(3), 387–416, 2008.

[MRT14] A. Mielke, S. Reichelt, and M. Thomas. Two-scale homogenization of nonlinear reaction-diffusion systems with slow diffusion. *Netw. Heterog. Media*, 9(2), 353–382, 2014.

[Mur78] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3), 489–507, 1978.

[Mur02] J. D. Murray. *Mathematical Biology. I*, volume 17 of *Interdisciplinary Applied Mathematics.* Springer-Verlag, New York, 3rd edition, 2002. An introduction.

[MuT97] F. Murat and L. Tartar. H-convergence. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser, Boston, MA, 1997.

[Muv11] A. Muntean and T. L. van Noorden. Homogenisation of a locally periodic medium with areas of low and high diffusivity. *European J. Appl. Math.*, 22(5), 493–516, 2011.

[Muv13] A. Muntean and T. L. van Noorden. Corrector estimates for the homogenization of a locally periodic medium with areas of low and high diffusivity. *European J. Appl. Math.*, 24(5), 657–677, 2013.

[NaR01] A. K. Nandakumaran and M. Rajesh. Homogenization of a nonlinear degenerate parabolic differential equation. *Electron. J. Differential Equations*, pages No. 17, 19 pp. (electronic), 2001.

[Neč67] J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques.* Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.

[NeJ07] M. Neuss-Radu and W. Jäger. Effective transmission conditions for reaction-diffusion processes in domains separated by an interface. *SIAM J. Math. Anal.*, 39(3), 687–720, 2007.

[Neu96] M. Neuss-Radu. Some extensions of two-scale convergence. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(9), 899–904, 1996.

[Ngu89] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3), 608–623, 1989.

[NiO01] B. Niethammer and F. Otto. Ostwald ripening: the screening length revisited. *Calc. Var. Partial Differential Equations*, 13(1), 33–68, 2001.

[NiO10] B. Niethammer and Y. Oshita. A rigorous derivation of mean-field models for diblock copolymer melts. *Calc. Var. Partial Differential Equations*, 39(3-4), 273–305, 2010.

[Nov08] A. Novick-Cohen. The Cahn–Hilliard equation. In *Handbook of differential equations: evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 201–228. Elsevier/North-Holland, Amsterdam, 2008.

[OnV07] D. Onofrei and B. Vernescu. Error estimates for periodic homogenization with non-smooth coefficients. *Asymptot. Anal.*, 54(1-2), 103–123, 2007.

[Pan97] A. Pankov. *G-Convergence and Homogenization of Nonlinear Partial Differential Operators*, volume 422 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1997.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[PeB08] M. A. Peter and M. Böhm. Different choices of scaling in homogenization of diffusion and interfacial exchange in a porous medium. *Math. Meth. Appl. Sci.*, 24, 119–136, 2008.

[PeB09] M. A. Peter and M. Böhm. Multiscale modelling of chemical degradation mechanisms in porous media with evolving microstructure. *Multiscale Model. Simul.*, 7(4), 1643–1668, 2009.

[Per12] J. Persson. Homogenization of monotone parabolic problems with several temporal scales. *Appl. Math.*, 57(3), 191–214, 2012.

[Pet09] M. A. Peter. Coupled reaction-diffusion processes inducing an evolution of the microstructure: analysis and homogenization. *Nonlinear Anal.*, 70(2), 806–821, 2009.

[Pie10] M. Pierre. Global existence in reaction-diffusion systems with control of mass: a survey. *Milan J. Math.*, 78(2), 417–455, 2010.

[PtR10] M. Ptashnyk and T. Roose. Derivation of a macroscopic model for transport of strongly sorbed solutes in the soil using homogenization theory. *SIAM J. Appl. Math.*, 70(7), 2097–2118, 2010.

[Rei14] S. Reichelt. Error estimates for nonlinear reaction-diffusion systems involving different diffusion length scales. *WIAS Preprint No. 2008*, 2014. Accepted in J. Phys.: Conf. Ser.

[RoS06] R. Rossi and G. Savaré. Gradient flows of non convex functionals in Hilbert spaces and applications. *ESAIM Control Optim. Calc. Var.*, 12, 564–614, 2006.

[RSS08] R. Rossi, A. Segatti, and U. Stefanelli. Attractors for gradient flows of nonconvex functionals and applications. *Arch. Ration. Mech. Anal.*, 187(1), 91–135, 2008.

[SaS04] E. Sandier and S. Serfaty. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, LVII, 1627–1672, 2004.

[Ser11] S. Serfaty. Gamma-convergence of gradient flows on Hilbert and metric spaces and applications. *Discrete Contin. Dyn. Syst.*, 31(4), 1427–1451, 2011.

[Sho97] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, volume 49 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[Smo94] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*, volume 258 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, New York, 2nd edition, 1994.

[SP*13] M. Schmuck, M. Pradas, G. A. Pavliotis, and S. Kalliadasis. Derivation of effective macroscopic Stokes-Cahn–Hilliard equations for periodic immiscible flows in porous media. *Nonlinearity*, 26(12), 3259–3277, 2013.

[Spa67] S. Spagnolo. Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore. *Ann. Scuola Norm. Sup. Pisa (3)*, 21, 657–699, 1967.

[Spa68] S. Spagnolo. Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597; errata, ibid. (3)*, 22, 673, 1968.

[Tar79] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass.-London, 1979.

[Tar09] L. Tartar. *The General Theory of Homogenization*, volume 7 of *Lecture Notes of the Unione Matematica Italiana.* Springer-Verlag, Berlin; UMI, Bologna, 2009. A personalized introduction.

[TB*03] U. Thiele, L. Brusch, M. Bestehorn, and M. Bär. Modelling thin-film dewetting on structured substrates and templates: Bifurcation analysis and numerical simulation. *Eur. Phys. J. E*, 11, 255–271, 2003.

[Tem88] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1988.

[Tim13] C. Timofte. Multiscale analysis of diffusion processes in composite media. *Comput. Math. Appl.*, 66(9), 1573–1580, 2013.

[Tur52] A. M. Turing. The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society (B)*, 237, 37–72, 1952.

[Vis04] A. Visintin. Some properties of two-scale convergence. *Atti Acad. Naz. Lincei Cl. Sci. Fis. MAt. Natur. Rend. Lincei (9) Mat. Appl.*, 15(1), 93–107, 2004.

[Vis07] A. Visintin. Two-scale convergence of some integral functionals. *Calc. Var. Partial Differential Equations*, 29(2), 239–265, 2007.

[Vis11] A. Visintin. Homogenization of a parabolic model of ferromagnetism. *J. Differential Equations*, 250(3), 1521–1552, 2011.

[Wlo82] J. Wloka. *Partielle Differentialgleichungen.* B. G. Teubner, Stuttgart, 1982. Sobolevräume und Randwertaufgaben. [Sobolev Spaces and Boundary Value Problems], Mathematische Leitfäden. [Mathematical Textbooks].

[Xin00] J. Xin. Front propagation in heterogeneous media. *SIAM Rev.*, 42, 161, 2000.